A STUDY OF THE BEHAVIOR OF INTERSTAGE QUEUES IN SERIES QUEUEING SYSTEMS WHERE STAGE MEAN SERVICE TIMES VARY WITH INTERSTAGE QUEUE LENGTH

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SUMMARY

This thesis is concerned with the study of the behavior of series queueing systems where the mean service time at each service station varies as a function of the size of the queue preceding that station. In this research, the mean service rate (which is the reciprocal of the mean service time) is assumed to increase with the number of units waiting for service. Two different functions are used to determine the state dependent service times. The variables controlled are:

1. The variance of the service time distribution,
2. The variance in the distribution of inter-arrival times at the first stage,
3. The position of the server in the sequence, and
4. The parameters in the model relating the mean service time to the size of the queue awaiting service.

The characteristics observed as dependent variables are:

1. The state probability distribution at each stage,
2. The mean waiting time at each stage, and
3. The inter-departure time distribution at each stage.

Monte-Carlo simulation methods, using GPSS II, are employed for each factor level combination (set of values for the controlled variables). Results are analyzed to determine the relationships existing among the exogenous and endogenous variables. An expression for the output distribution at the first stage when the arrivals are Poisson and the services are exponential is derived for a particular state dependency relationship.

CHAPTER I

INTRODUCTION

Queueing theory represents one of the main areas of operations research, having as it does in its exhaustive repertory at least 2000 references (53). It is a chief concern in system design, since it offers an approach to the solution of problems of congestion, traffic flow, inventory systems, etc. Waiting line theory (an alternate name for queuing theory) is based upon the mathematical theory of probabilities.

An area of considerable interest in such classes of problems as those embraced by the theory of waiting lines is the one that studies the characteristics of series queues by which is meant a service facility consisting of several service stations through which each serviced unit must pass in a specified order. Ready applications may be found in queueing problems faced by department stores, college registrations, certain traffic flow problems, factory assembly lines, production systems, etc.

Consider a series queueing system as shown in Figure 1. The symbol S_j denotes the j th server ($j=1,2,3,-----,N$), and Q_j denotes the jth interstage position where a queue may form. This could be imagined to be an assembly line in which each unit must pass through several assembly operations before it comes out of the assembly line as a finished product. Such a network of service centers, all of which are theoretically equally loaded, may experience very considerable short term variations in load that result in serious system imbalance. The disparity in the state be-

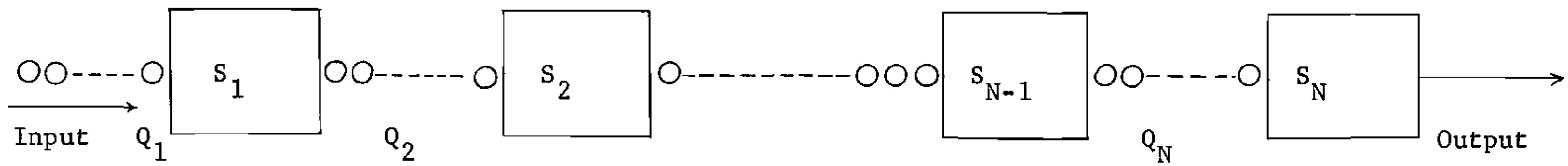

Figure 1. A Tandem Queueing System

tween the different centers becomes very large, with some queues becoming very large while other centers remain idle. While it is true that the work in process in an assembly line is not uniformly distributed between the various work centers, experience suggests that the disparity of the distribution in an actual shop is rarely as great as that predicted by a queueing model which has invariant service time distribution parameters. This indicates that in an actual shop some kind of adjustments are made in order to improve the short-term balance between service centers.

Conway and Maxwell (11) have listed three types of adjustments that can be made in a production shop in order to better balance work-loads.

(A) External. The sequence of jobs released to the shop can be controlled so as to balance the workloads of different service centers.

(B) Network Discipline. The flexibility of network discipline includes the sequence of jobs to centers. Priority can be given to jobs which move to the next center which needs work for their next operation.

(C) Service Center. The service center reacts by altering the rate at which it provides service. When there is a large backlog of work, the center operates "faster" than when the backlog is small or non-existent. An increase in service rate could mean an increase in effort on the part of the worker, compromise on the quality of the product, use of indirect labor to assist some work elements, etc.

This research is concerned with the adjustment of type three given above where there is no external control and the service is represented by a fixed series of servers. We assume that the increase in service rate is related to the number of units waiting in the queue before each service station.

In this study, two models are considered to represent the variation in the mean service rate at each stage. The system is to be balanced, meaning that the base mean service rate* denoted by μ, will be the same at each stage. At the first stage, the base mean service rate is equated to the base mean arrival rate, λ , where λ is the reciprocal of the mean time between arrivals.

Since the departure from one stage represents the arrival to the next stage, the inter-departure time distribution at one stage represents the inter-arrival time distribution at the next.

Nature of the Present Investigation

Objectives and Purposes

The objective of this research is to study the behavior of series or sequential queueing systems where the mean service times at each stage varies as a function of the inter-stage queue length. In this study, the behavior of a state-dependent series queueing system is characterized by:

1. The distribution of the nature of formation of queues at various inter-stage positions (the state probability distribution at each stage),

2. The expected waiting times of any unit at each stage, and

3. The time-between-departure distribution (output distribution) at each stage.

The variables controlled are:

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*The base mean service rate is the mean rate at which service is provided when the server works at a nominal rate.

1. The variance of the service time distribution,

2. The variance in the distribution of inter-arrival times at the first stage,

3. The position of the service station in the sequence, and

4. The parameters in the model relating the mean service time to the size of the queue awaiting service at each station.

The purpose of this research is, of course, to obtain results which are applicable to real world problems. System designers responsible for the design of service or processing systems have been without much information to aid in the design of such facilities.

Method of Attack

The experiment is designed as a four factor factorial experiment. The four controlled variables defined above are taken to be the factors varied in the study. System simulation employing Monte-Carlo methods is used for each combination of the factors (each "treatment").

Form of Results

Chi-Square goodness of fit test is used to compare the results obtained from this study with the hypothesized results. The characteristics observed as dependent variables are graphed against the input variables to show the estimated relationships. In addition, tabular results are also displayed.

Scope, Limitations, and Assumptions

The simulation model developed is not restricted to any particular distribution for the arrival and service processes. The results obtained by simulation are limited only to the two models for service time variation studied. The arrival process for jobs into the system is assumed to be a stationary random process. The service times at each service center are assumed to have a known probability density function. Infinite queues are allowed between stages. No server is allowed to be idle if there are units in its queue. A unit is assumed to be available at a service center for service as soon as its preceding operation is completed. The queue discipline is first come first served. No balking, reneging, feedback or other complications are allowed. When the service of a unit starts, its mean service rate is calculated as a function of the number of units in the queue preceding that service stage. It is assumed that the rate of service does not increase if more units join the queue when the service of the unit is in progress.

CHAPTER II

LITERATURE SURVEY

The pertinent literature may be classified as that dealing with (1) the behavior of queues in series queueing systems, and (2) with assembly line balancing problems in particular.

Most of the analytical work on queues with a number of facilities in series has been restricted to Poisson arrivals and exponential service times with phase type servicing. Phase type servicing refers to channels in series without the provision for waiting before any service channel except the first one. A new item is admitted into service after the previous item completes all the phases. Study of the output of one channel has received considerable attention beginning in 1956. In the study of a tandem queue the output from one channel comprises the input into the subsequent channel.

The steady-state departure process from a Poisson-Exponential service stage has been studied by Burke (3), Reich (49, 50) Cohen (7), and Finch (20). Burke proved that the steady state output of a service channel, with Poisson input and negative exponential service time is itself Poisson, i.e., the inter-departure intervals are exponentially distributed and are independent random variables. Working independently, Reich established that the flow times through Poisson-Exponential servers in tandem are independent. He also proved that for a single channel queue, Poisson arrivals and departures imply an exponential service time distribution or a step function at zero. Finch established that Burke's Poisson departure result holds only when an infinite queue length is allowed. He also determined that independent exponential service times and unbounded queue length are necessary and sufficient conditions for independent inter-departure intervals and the independence of the queue length left by a departing unit from the interval since the previous departure.

Chang (6) gave a derivation for determining the output of a single channel service stage with general independent arrival intervals and general independent service times. Chang's work is based on the Wiener-Hopf integral equation for the waiting time distribution of the system which was obtained by Lindley (33).

Good (23), perhaps, was among the first to study the number of units in a tandem queueing system. O'Brian (47) studied the case of two channels in series with Poisson input and exponential service times and gave expected queue lengths and expected waiting times. R. R. P. Jackson (31, 32) extended O'Brian's work in the study of tandem queues in the steady state with Poisson assumptions (with different negative exponential service distributions for each stage) and he gave the probability of various numbers of items at the different stages, and the waiting time distribution in each phase. J. R. Jackson (29, 30) studied a tandem queuing system wherein a unit arrives at different phases with different probabilities.

The ergodic properties of two queues in series have been studied by Akaike (1) and Sacks (54). Nelson (45) assumdd different exponential service time distributions at each stage in a series queueing system and derived the probability of waiting longer than a given time at all channels. DeBaun and Katz (15) assumed Chi-square approximation to the sum of exponentials, to simplify Nelson's computations.

Hunt (27) studied the utilization for the queues with Poisson arrivals and exponential service times for several queue disciplines:

1. An infinite queue allowed between each channel,

2. Finite queue at the second and succeeding stages,

3. A zero queue except at the first channel, and

4. A zero queue with no vacant facilities (the entire line moves as one unit).

Morris (43) treated some materials handling problems as queueing networks. Conolly (8, 9, 10) has examined simple queues using a difference equation technique.

Some more recent papers have treated non-Poisson series queues. Ghosal (22) has studied a two stage queue assuming a Poisson input and a Gamma service distribution at the first stage and an exponential service distribution at the second stage. Suzuki (55) has dealt with two queues in series with infinite inter-stage queue allowed, and Chesbrough (5) has developed a theory of output behavior for the important application of queueing network analysis.

Hillier and Boling (25) have considered a queueing system consisting of N service channels in series where each channel has an exponential or Erlang holding time and (except for the first channel) a finite queue, and where the input process is such that the first queue is never empty. The measures considered are the steady state mean output rate and the mean number of customers in the system (excluding the first queue).

In spite of the large amount of literature that is available on tandem queues, very little has been done to solve queueing problemswwhen the arrival and service distributions are state dependent. Jose'de la Cruz, Jamie Tolra and others (17) did a simulation study of a tandem queue with state dependent service times. They published the result for six combinations of arrival and service distributions. In another paper, Conway and Maxwell (11) presented some results where:

1. The service rate is dependent on the length of the queue preceeding that service station.

2. Arrival rate is state dependent, and

3. Arrival rate and service rate are both state dependent.

Nelson (46) has published the results of a series of simulation experiments with a two server queueing network model, and Harris (25) has generalized the standard M/G/1 queueing system so that the service time parameter becomes a stochastic process indexed on the length of the queue at the moment service is begun.

CHAPTER III

DESIGN OF THE EXPERIMENT

The three fundamental problems of interest in a sequential queueing system are:

1. The structure of formation of queues between the successive stages,

2. The influence of the service time distribution at each stage on the inter-departure time distribution (output distribution) from the corresponding stage, and

3. The nature of the waiting time distribution between stages.

This research is oriented towards studying the above three factors against varying input parameters.

This research is designed as a four-factor factorial experiment as follows:

1. The time between arrivals at the first stage of the series queueing system follows the Erlang distribution with parameter K_1. Its density function can be recognized as

$$f(t) = \frac{K_1 \lambda}{(K_1-1)} \quad (K_1 \lambda\, t)^{K_1-1}\, e^{-K_1 \lambda t} \quad ; \; t > 0 \qquad (1)$$

$$0 \; ; \; \text{otherwise}$$

where λ is the mean arrival rate

so that

$$E(t) = \frac{1}{\lambda} \qquad (2)$$

$$V(t) = \frac{1}{K_1} \left(\frac{1}{\lambda}\right)^2 \tag{3}$$

$$\text{Std Dev }(t) = \frac{1}{\sqrt{K_1}} \quad E(t) \tag{4}$$

As the parameter K_1 is increased the first moment, $E(t)$, remains unchanged while the standard deviation decreases as the reciprocal of $\sqrt{K_1}$. The parameter K_1 is one of the factors varied in the study.

2. The number of stages in the sequential process is 15 (obviously the characteristics of a system having any number of stages less than 15 can be studied from these results).

3. The base mean service rate, μ, is the same at each service stage and is equal to the mean arrival rate λ . The service time distribution belongs to the Erlang family with parameter K_2, and K_2 is another factor varied in the study.

4. Adjustments are made to the mean service rate at each stage depending on n, the number of units preceding the particular service station. Each stage with the queue associated with it is considered to represent a "system." Thus n is the sum of the number in queue preceding a stage and the number in service at that stage. As infinite queues are allowed to form between stages, n for each system may vary from 0 to ∞. It should be pointed out that even though infinite queue buildup is a tacid assumption in the initial stages of study, this assumption led to an evaluation of the optimum inventory space between stages that would make this assumption irrelevant for the later stages of the study.

If the mean rate of service at a stage is equal to the mean rate of departures from the previous stage ($\rho = \frac{\lambda}{\mu} = 1$) we know that the behavior of such a line will be unstable unless some influence is exercised. Since in real life the space between two consecutive stages (in-process inventory space) is limited, we should either withdraw the excess units between stages or allow the line to be "blocked" when the number of units between two stages exceeds the maximum allowable space. In this research it is assumed that the service rate varies as a function of the number in the system so as to limit the maximum queue length. This mean service rate can be represented as

$$\mu_n = x_n \mu , \tag{5}$$

where X_n is a function of n.

It was pointed out in the literature survey that Jose'de la Cruz, et al., (17) have made a simulation study of a tandem queue with state dependent service times. Their study is limited to a two-stage queue and to only one model for service time variation. Further they have assumed only one value of the parameter in the model in studying the system. In this study, two models are presented in considering the variation at each stage with varying values of the parameter in each model.

The two models are:

<u>Model I.</u>

when

$$\mu_n = X_n \mu$$

$$\rho_n = \frac{\lambda}{\mu x_n} = \frac{1}{X_n} \quad (\text{where } \lambda = \mu) \tag{6}$$

we assume that ρ_n varies exponentially with n so that

$$\rho_n = \frac{1}{X_n} = Be^{-(n-1)} + A \tag{7}$$

where A and B are such that $A + B = 1$.

Thus for $n = 1$, $\rho_n = A + B = 1$, and as $n \to \infty$, $\rho_n \to A$.

In Figure 2 the variation of the "utilization coefficient" ρ_n is shown against the number in the system for varying values of A and B.

Model II.

$$\mu_n = X_n \mu \qquad \text{where } X_n = n^c \tag{8}$$

The value of c can be controlled to suit the particular system. One may observe that when $c = 0$ the model represents a system with no state dependency relationship. When $c = 1$, the service rate is directly proportional to the system state. One can also consider values of c less than zero to imitate a system where the service center slows down as the work accumulates. A good example of the latter case is when a clerk in an office tends to work slower as the amount of paper work on his table piles up. The values of ρ_n are plotted for this model against n in Figure 3.

The experimental design is summarized in reference form in Figure 4. The input parameters and the experiment code symbols used in the table are explained below.

The Erlang parameter K_1 in the arrival distribution is varied as 1, 5, 10, and 20. K_2 in the service distributions is given the values 1, 5, 15, 20, and 25 respectively. The values of A in model I are 0.9, 0.8, 0.7, 0.6, and 0.5.

Parameter c in model II takes the values 0.3, 0.5, and 1.0. Experiments are performed for all possible combinations of K_1, K_2, A and

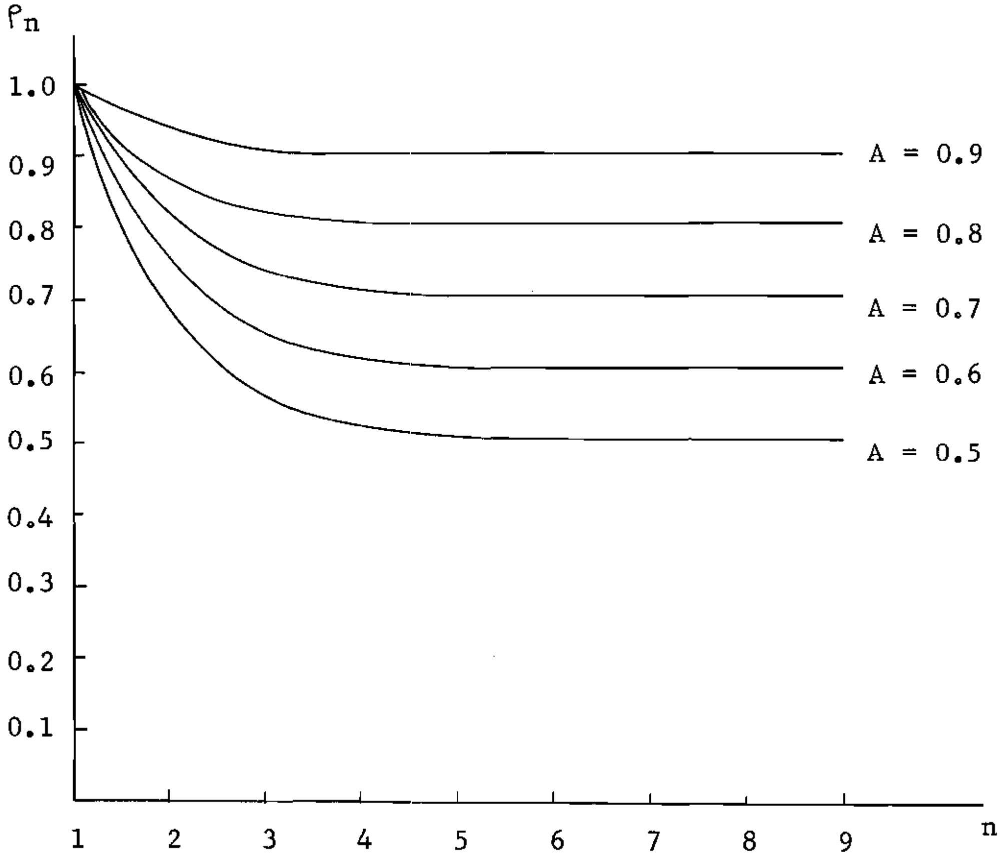

Figure 2. Variation of ρ_n Against n for Varying Values of A in Model I

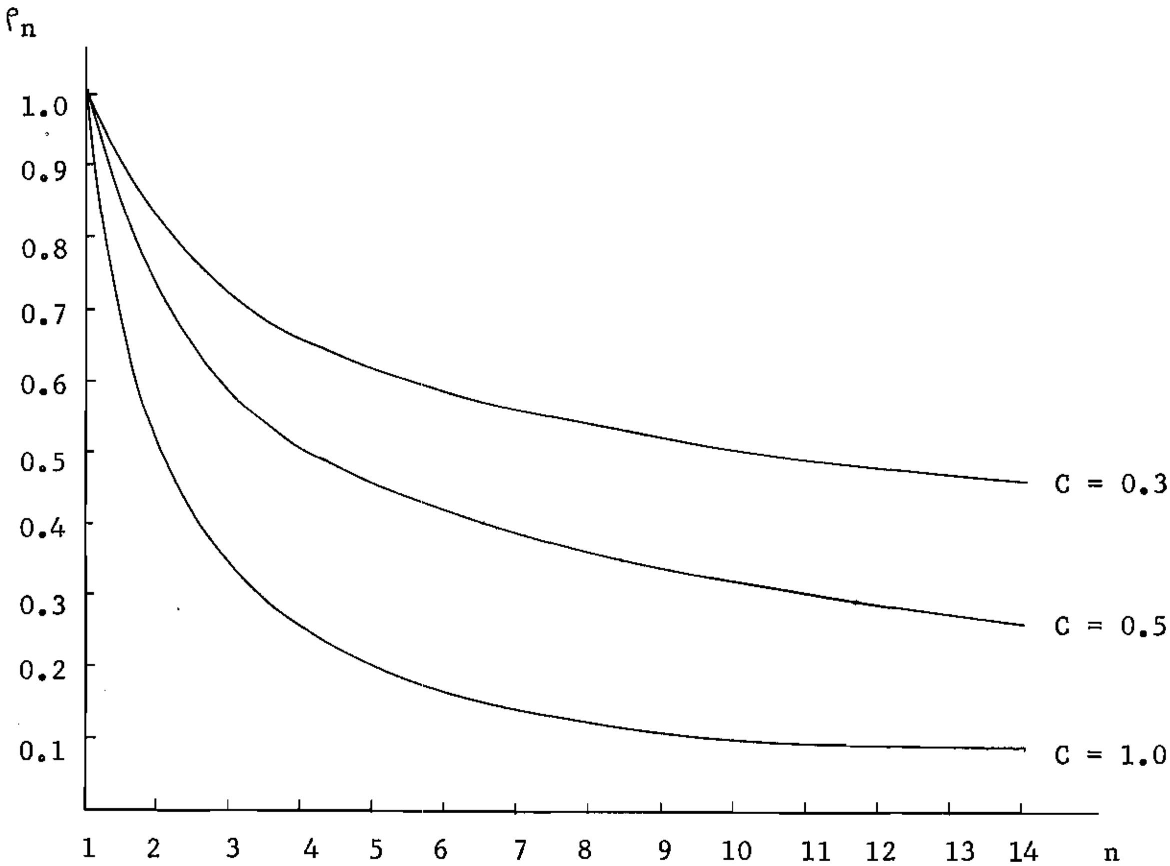

Figure 3. Variation of ρ_n Against n for Varying c in Model II

		Model I $\rho_n = Be^{-(n-1)} + A$				
		K_2 in service time distribution				
		1	5	15	20	25
K_1 in arrival distribution	1	AA1	AB1	AC1	AD1	AE1
	5	BA1	BB1	BC1	BD1	BE1
	10	FA1	FB1	FC1	FD1	FE1
	20	DA1	DB1	DC1	DD1	DE1

		Model II $\rho_n = \frac{1}{n^c}$				
		K_2 in service time distribution				
		1	5	15	20	25
K_1 in arrival distribution	1	AA2	AB2	AC2	AD2	AE2
	5	BA2	BB2	BC2	BD2	BE2
	10	FA2	FB2	FC2	FD2	FE2
	20	DA2	DB2	DC2	DD2	DE2

Figure 4. Experimental Design

c. The first and second letters in the experiment code symbol designates the combination of the arrival and service processes used for that experiment as has been shown in Figure 4. The third symbol in the experiment code gives the number of the model used for the run. The values of the parameter in model I or model II, and the interstage position can be combined with the experiment code symbol without much difficulty. For example

AC_1 - 0.5 - 12 represents the run AC_1 with A = 0.5 and indicates that we are referring to the 12th interstage position.

Analytical Results

Harris (25) has given the derivation of the steady-state probabilities of a single channel queue with Poisson arrivals and exponential service times where the service rate for a particular customer is a function of the number in the system at the moment the customer enters the service channel. It is summarized as follows for model II with c = 1.

We have $$\mu_n = n\,\mu \tag{9}$$

Thus the density function of the service time when there are n in the system at the start of service to a particular customer is

$$g(t_n) = nue^{-n\mu t_n}\ ;\ t_n > 0 \tag{10}$$

The distribution function is

$$G(t_n) = 1-e^{-n\mu t_n} \tag{11}$$

Let $k_{ij} = \Pr\left\{ i \text{ arrivals during service time} \mid j \text{ in system when service began.}\right\}$

The departure process of such a system has an imbedded Markov chain the transition matrix of which is

$$P = p_{ij} = \begin{bmatrix} k_{01} & k_{11} & k_{21} & k_{31} \\ k_{01} & k_{11} & k_{21} & k_{31} \\ 0 & k_{02} & k_{12} & k_{22} \\ 0 & 0 & k_{03} & k_{13} \\ - & - & - & k_{04} \end{bmatrix}$$

Harris has proved that a state dependent queueing system with exponential inter-arrival times and services is ergodic when $\rho_1 = 1$ and $\rho_n < 1$, $n > 1$, independent of the form of μ_n. He has further shown that the system is not ergodic when $\lambda/\mu = \rho = \rho_1 > 1$. If $\pi_n = \Pr\{$ n customers in system at an arbitrary point of departure in steady state$\}$, $\{\pi_i\}$ could be determined by iteration on

$$\pi_j = \pi_0 k_{j1} + \pi_1 k_{j1} + \pi_2 k_{j-1,2} + \text{-----} + \pi_{j+1} k_{0,j+1} \tag{12}$$

where

$$\left.\begin{aligned} k_{ij} &= \frac{1}{i!}\int_0^\infty (\lambda t)^i e^{-\lambda t}\, dG(t_j) \\ &= \frac{1}{i!}\int_0^\infty (\lambda t)^i e^{-\lambda t}\, j\mu e^{-j\mu t}\, dt \\ &= \frac{\lambda^i \mu j}{(\lambda+\mu j)^{i+1}} \end{aligned}\right\} \tag{13}$$

From the above,

$$\pi_0 = \pi_0 k_{01} + \pi_1 k_{01}, \quad \pi_1 = \pi_0 \left[(1-k_{01})/k_{01}\right]$$

But $$k_{o1} = \mu/\lambda+\mu$$

And hence
$$\begin{aligned}\pi_1 &= \left\{\left[1 - \mu/(\lambda+\mu)\right]/\mu/(\lambda+\mu)\right\}\pi_0 \\ &= \lambda/\mu\ \pi_0 \\ &= \rho\pi_0 \quad \text{Where } \rho = \rho_1 = \lambda/\mu \quad \text{and} \quad \rho_n = \lambda/(n\mu) \end{aligned} \tag{14}$$

The remaining π_j are obtained by iteration on (12).

Output Distribution

The output distribution for the above cage (when the arrivals follow a Poisson distribution and service times follow an exponential distribution, the parameter in the service time distribution being directly proportional to the number in the system at the start of service) is derived as follows:

If a departure occurs at time $t = 0$, the next departure can occur in one of the following two mutually exclusive and collectively exhaustive ways:

(A) The departing unit leaves an idle system behind. In this case the time for the next departure to occur, t_{bd}, consists of t_{ba}, the time till next arrival and $t_{s|\mu}$ the service time for the arrival at a rate μ. $h_o(t_{bd})$ is the distribution of the sum of two random variables t_{ba} and $t_{s|\mu}$ (which are exponentially distributed with parameters λ and μ respectively).

Thus $h_o(t_{bd})$ is the convolution of $g(t_{ba})$, the arrival distribution and $v(t_{bd|\mu})$, the service distribution.

Or, $$h_o(t_{bd}) = g(t_{bd}) * v(t_{bd|\mu}). \tag{15}$$

Using Laplace transforms,

$$\mathcal{L}\left[h_o(t_{bd})\right] = \frac{\lambda}{s+\lambda} \cdot \frac{\mu}{s+\mu} \tag{16}$$

Let

$$\frac{\lambda\mu}{(s+\lambda)(s+\mu)} = \frac{P}{(s+\lambda)} + \frac{Q}{(s+\mu)}, \tag{17}$$

and solving for P and Q,

$$P = \frac{\lambda\mu}{(\mu-\lambda)}$$
$$Q = -\frac{\lambda\mu}{(\mu-\lambda)} \tag{18}$$

and therefore,

$$\mathcal{L}\left[h_o(t_{bd})\right] = \left(\frac{\lambda\mu}{\mu-\lambda}\right)\left(\frac{1}{s+\lambda} - \frac{1}{s+\mu}\right) \tag{19}$$

Taking the inverse transform

$$h_o(t_{bd}) = \left(\frac{\lambda\mu}{\mu-\lambda}\right)\left(e^{-\lambda t} - e^{-\mu t}\right) \tag{20}$$

and it is noted that $h_o(t_{bd})$ occurs with a probability π_o, the probability of an idle system.

(B) The departing unit leaves the system in state n, $n > 0$. Thus the next unit which enters service is served at a rate $\mu_n = n\mu$.

The density function of the output distribution when there are n in the system is

$$g_n(t_{bd}) = n\mu e^{-n\mu t} \tag{21}$$

$g_n(t_{bd})$ occurs with probability π_n.

The inter-departure time distribution from the system, $g(t_{bd})$, is:

$$g(t_{bd}) = \pi_0\left(\frac{\lambda\mu}{\mu-\lambda}\right)\left[e^{-\lambda t} - e^{-\mu t}\right] + \sum_{n=1}^{\infty} \pi_n\, n\mu e^{-n\mu t} \tag{22}$$

For the special case where $\lambda = \mu = 1$ equation (2) becomes

$$g(t_{bd}) = \pi_0 t e^{-t} + \sum_{n=1}^{\infty} \pi_n\, n e^{-nt} \tag{23}$$

Since,

$$\lim_{\mu \to \lambda} \left\{ \left(\frac{\lambda\mu}{\mu-\lambda}\right)\left[e^{-\lambda t} - e^{-\mu t}\right]\right\} = t e^{-t}$$ (using l'Hospital's rule).

The inter-departure events are shown diagramatically in Figure 5.

$$\begin{aligned} E(t_{bd}) &= \int_0^{\infty} \pi_0 t^2 e^{-t}\,dt + \sum_{n=1}^{\infty} n\pi_n \int_0^{\infty} t e^{-nt}\,dt \\ &= 2\pi_0 + \sum_{n=1}^{\infty} \frac{\pi_n}{n} \end{aligned} \tag{24}$$

$$\begin{aligned} E(t_{bd}^2) &= \int_0^{\infty} \pi_0 t^3 e^{-t}\,dt + \sum_{n=1}^{\infty} n\pi_n \int_0^{\infty} t^2 e^{-nt}\,dt \\ &= 6\pi_0 + 2\sum_{n=1}^{\infty} \frac{\pi_n}{n^2} \end{aligned} \tag{25}$$

$$V(t_{bd}) = E(t_{bd}^2) - \left[E(t_{bd})\right]^2 \tag{26}$$

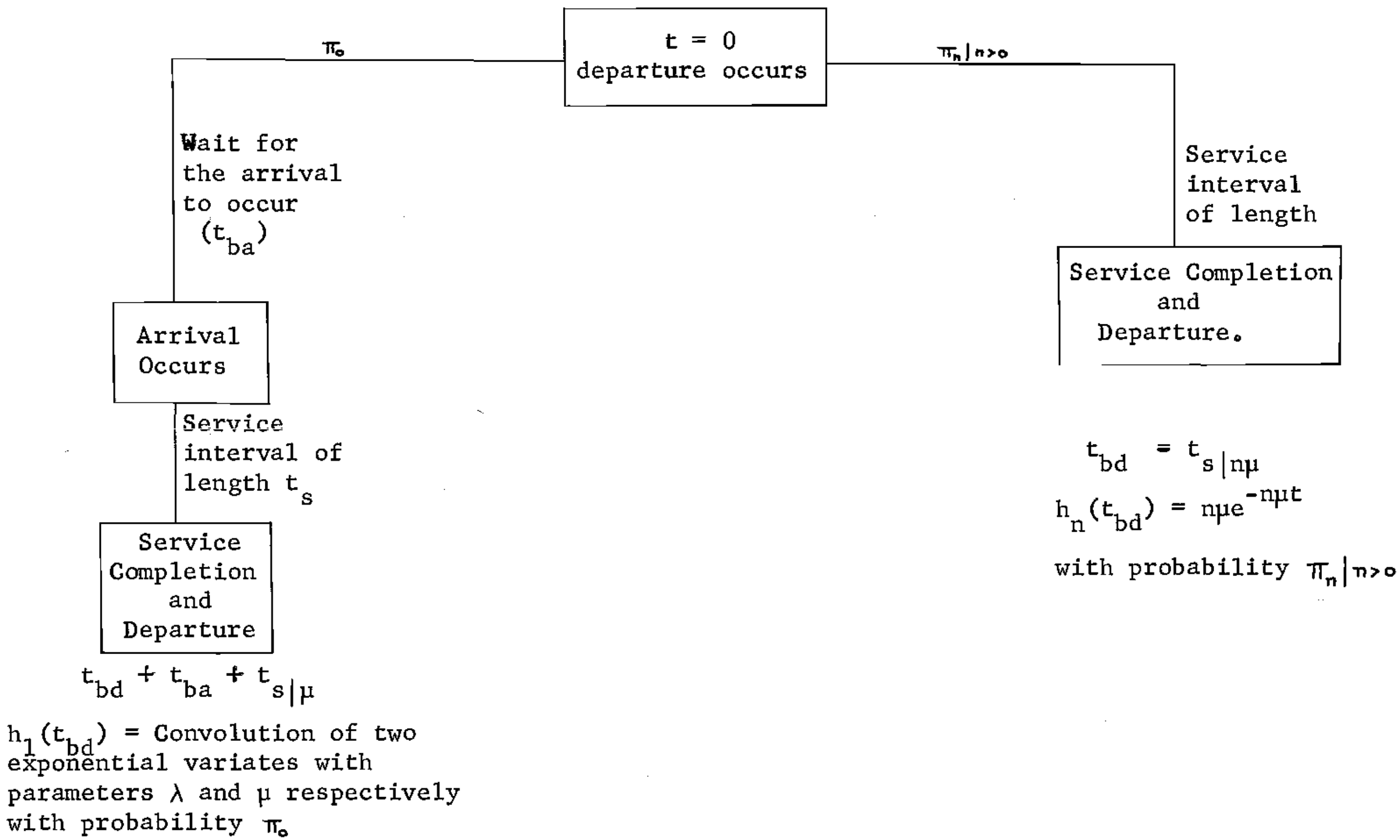

Figure 5. Inter-departure Events for $M|M_n|1$.

CHAPTER IV

THE SIMULATION

It is generally true that the analysis of quite simple stochastic processes often leads to intractable mathematics. In such circumstances, it is nowadays customary to resort to digital computer simulation. With the advent of computers and with their increasing availability, computer simulation is becoming a widely used technique. The practice of simulation does, however, require great care; in particular it is necessary for the user to convince himself that his computer program is free from fault and that it provides a valid model.

Monte-Carlo simulation shall be used to analyze the system developed in the previous chapter. The important properties of the system shall be derived and analyzed so that it is possible to experiment with alternate courses of action. Simulation is particularly suitable for use in queueing theory. Since computer programs form the crux on which simulation has been built, great attempts have been made to develop new computer languages specifically designed to meet the needs of various situations; to quote Naylor (44) they have been designed with the following objectives in mind.

1. To produce a generalized structure for designing simulation models.

2. To provide a rapid way of converting a simulation model into a computer program.

3. To provide a rapid way of making changes in the simulation model that can be readily reflected in the machine program.

4. To provide a flexible way of obtaining useful outputs for analysis.

The simulation of queueing systems belongs to a family of problems dealing with situations which profit by the use of a discrete time variant type of a language. Probably the best and the best known among such languages is the GPSS, the General Purpose Systems Simulator (28).

The ease of computation and analysis as well as the built-in characteristics of a language such as the GPSS are of immense use. Hence the basis for selecting GPSS as the language of description of the system. This language is suitable for simulating waiting line type of situations, where it is required to compute the queue statistics and different output distributions of interest.

The Simulator

GPSS II, which was developed by R. Efron and G. Gordon (28), is a "block diagram" oriented language; i.e., each block belongs to a fixed set of block types. Each block represents some specific operation performed in the simulation process. The block diagram for the series queueing system being simulated is shown in Figures 6a and 6b. The reader who is not familiar with the GPSS II terminology is referred to Appendix A for a brief resumé on the block types used.

The ORIGINATE block (block 1, Figure 6a) creates transactions at stochastic time intervals according to the function (FN1) with a mean rate of one transaction every 100 time units. These transactions repre-

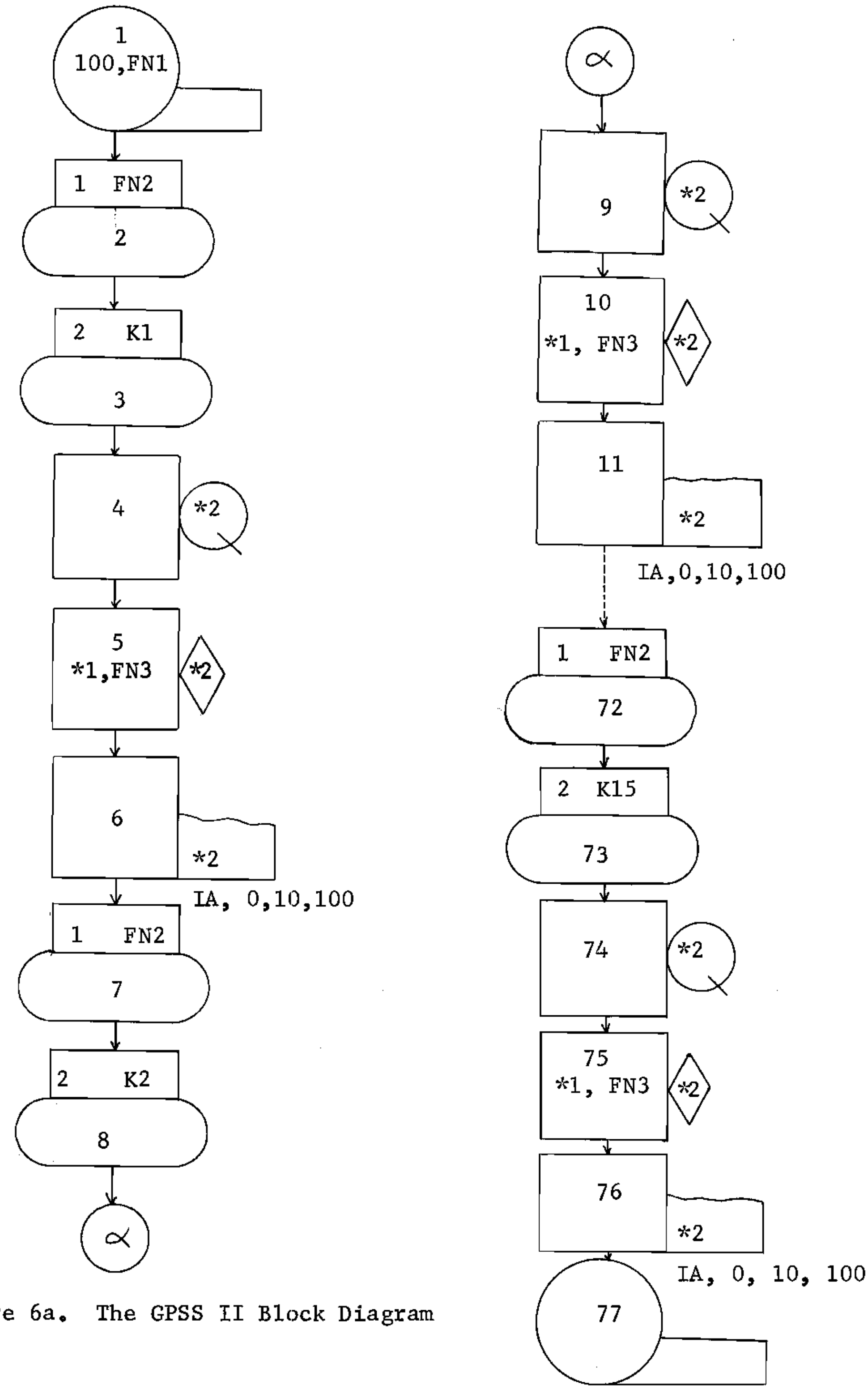

Figure 6a. The GPSS II Block Diagram

sent units entering the system. Since the system being simulated consists of 15 sequential service stations, each transaction is passed through all these stages in a specified order before it is removed from the system at the TERMINATE block (block 77). System statistics are recorded at each service stage using the block types QUEUE, HOLD, and TABULATE. Blocks two to six simulate the first stage of the sequential process. Block two assigns a variate of the function (FN2, which is the service time distribution at each stage of the system under consideration, with a mean rate of $\mu = \lambda = (1/100)$ to the parameter field one of the transaction entering the block. This service time has to be modified using the function (FN3, which defines the relationship between the mean service time and the number in the queue) to give the actual service time for the customer under consideration. The above modification on the service time is done at the HOLD block (block five) where the customer actually gets serviced. If, however, the customer finds the facility occupied when he tries to enter it, he waits in line (block four, the QUEUE block) for his turn to get served. After he is served at one service facility he enters the next stage. The above process repeats at each of the 15 stages before the customer leaves the system. Block six is a TABULATE block. In this case it records the inter-departure time distribution from stage one of the sequential waiting line system. The queue number in block four, the facility number in block five, and the table number in block six are specified using the technique of indirect specification as shown. This is achieved by assigning values one, two, three, ---- 15, to the parameter field two of the transaction at the corresponding service stage. The functions of blocks seven to 76 are obvious. Figure 6b gives the block diagram for tabulating the state probability distribution at each stage. The queue length at each stage of the system is measured at every 50 time units and tabulated using block 93.

Validation

In verifying the validity of a model, what we do is to prove (or to disprove) that the model does represent the system it simulates. The theoretical results developed in Chapter IV, for the first stage of a state dependent series queueing system, will be used to check the authenticity of the simulation model.

The fourth column of Table 1 gives the expected frequency in state n_q, while the fifth column gives the corresponding values observed from a simulation run. Measures have been taken to ensure that steady state was represented. The calculations for a Chi-square goodness of fit test need no explanation. The deviation of observed values from those expected is not significant using this test. The computer model was also checked for the case where there is no state dependency relationship and where the input to the first stage is Poisson and the services at each stage is exponential. From the theory of queueing the output from each stage of a series queueing system is Poisson with the same parameter as the input distribution.

Based on the above observations, the simulation model is accepted to represent the system under study.

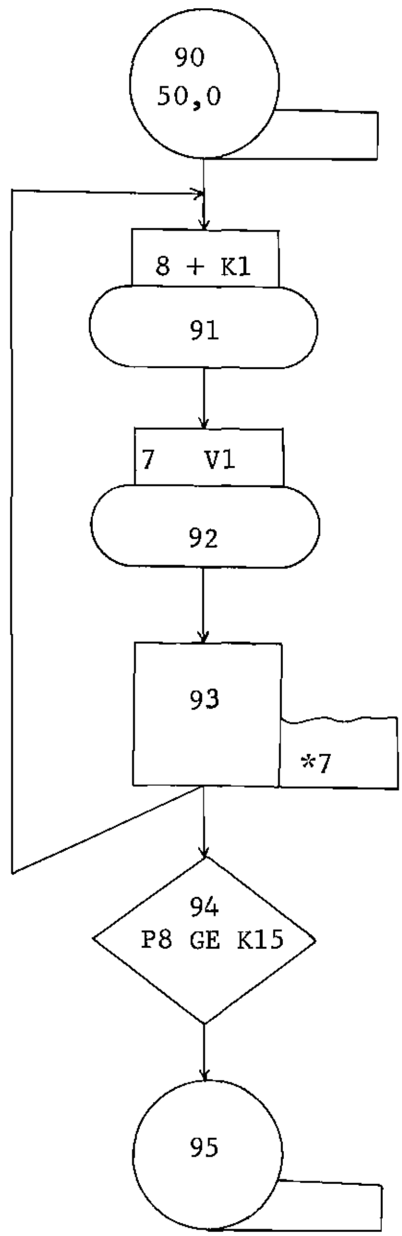

Figure 6b. Block Diagram for Obtaining the State Probability Distribution at Each Stage

Table 1. Comparison Between Theory and Simulation of the State-Probabilities for the case when $\mu_n = n\mu$ in Model II

Number in System n	Number in Queue n_q	Probability of n in System π_n	Expected Frequency in state n_q E	Observed Frequency in state n_q 0	$\frac{(E-0)^2}{E}$
0	0	.2764	6678.27	68311	3.49
1		.2764			
2	1	.2073	2506.43	2462	0.79
3	2	.1229	1485.29	1444	1.15
4	3	.0624	754.25	736	0.44
5	4	.0291	351.83	365	0.49
6	5	.0132	159.08	152	0.32
7	6	.0064	77.31	69	0.89
8	7	.0032	38.54	24	5.49
9	8	.0017	20.86	21	0.00
10	9	.0010	12.59	10	0.53
				χ^2 (9 degrees of freedom)	13.59

CHAPTER V

ANALYSIS OF RESULTS

Study of the Output Distribution at Each Stage

Two important measures of interest of most random variables are the first moment about the origin (mean) and the square root of its second moment about the mean (standard deviation).

From the results it is observed that the mean of the output distribution (inter-departure time distribution) from any stage is equal to the mean input to that stage. This result is expected of a system where the service rate is greater than or equal to the arrival rate, which is true in the system under study. Further, since in a sequential queueing system the output from one stage constitutes the input to the subsequent stage, the mean of the inter-departure time distribution from each stage is found to be identically equal to the mean input to the first stage. However, the standard deviation (stdndev.) of the output distribution varies from stage to stage depending upon the input parameters. Figures 7 to 10 show the above variation for different input and service time distributions. Only one value of parameter A in model I (A = 0.9) and one value of c in model II (c = 0.3) are considered in plotting the curves. It is worth mentioning that though the curves are shown to be continuous, values should be read from the graphs only against discrete stage numbers. The standard deviation of the inter-departure time distribution at the first stage is tabulated for all combinations of input

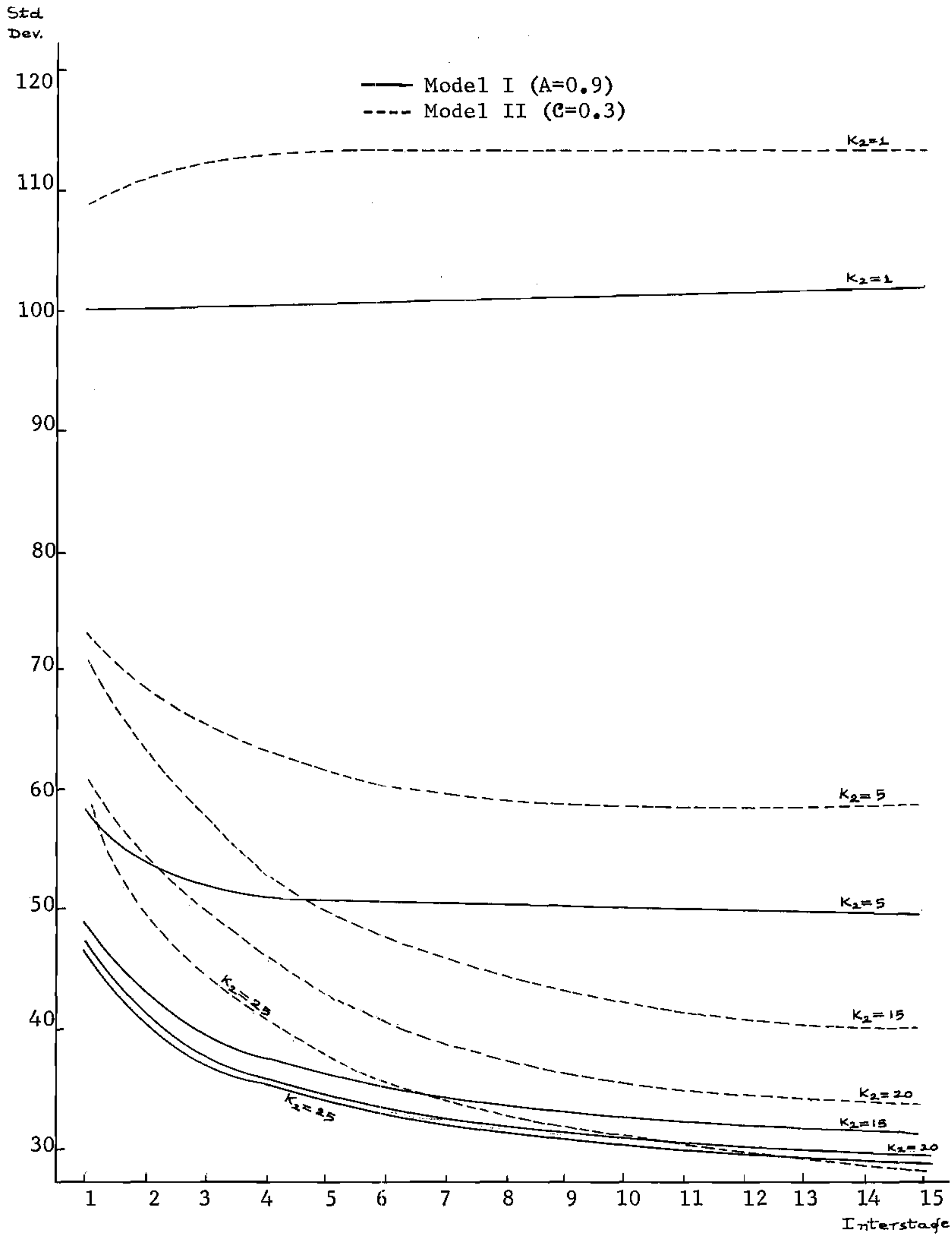

Figure 7. Standard Deviation of the Inter-departure Time Distribution Shown Against the Inter-stage Position - Poisson Arrivals (K_1 =1)

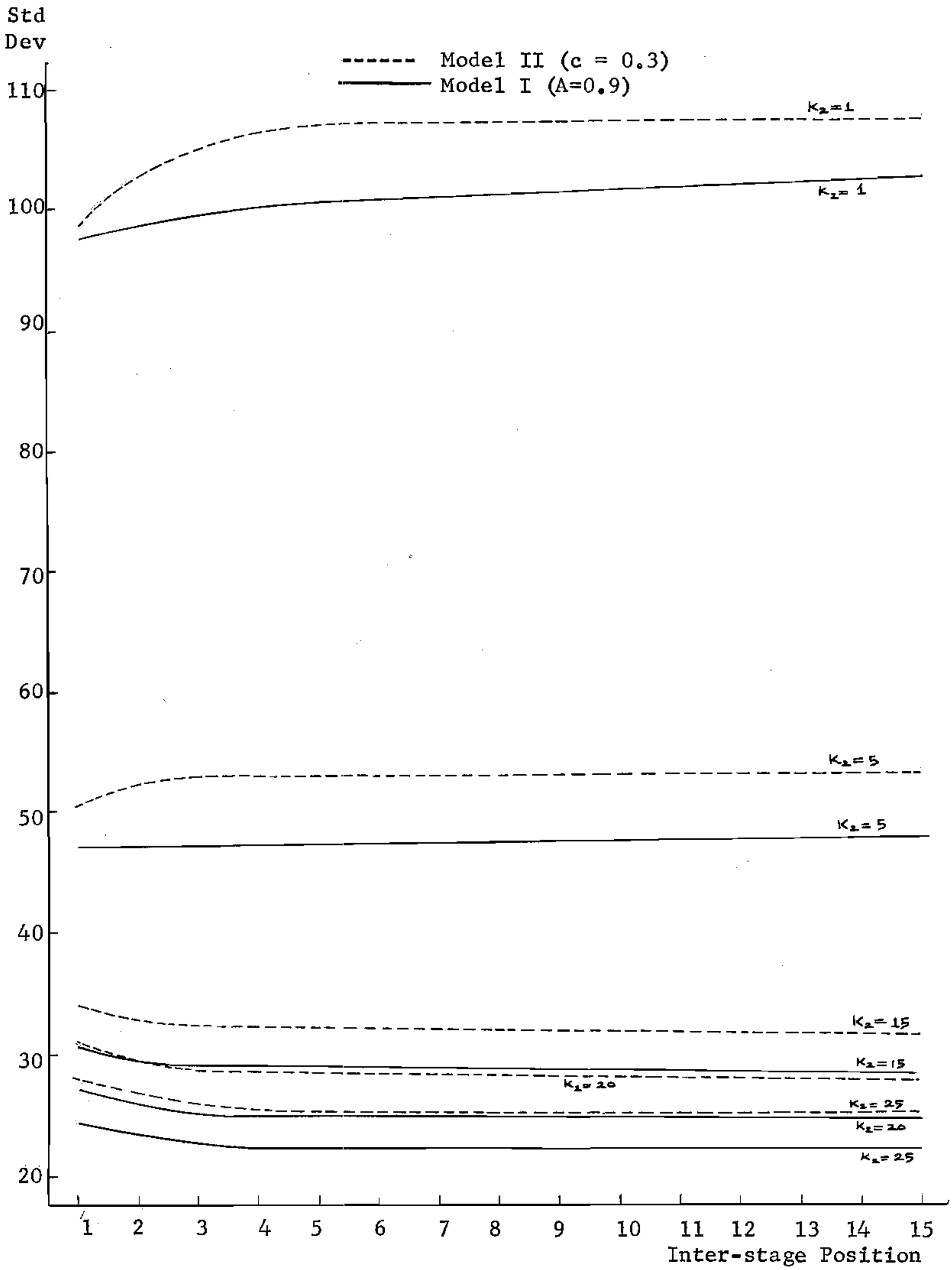

Figure 8. Standard Deviation of the Inter-departure Time Distribution Shown Against the Inter-stage Position - Erlangian Arrivals (K_1 = 5)

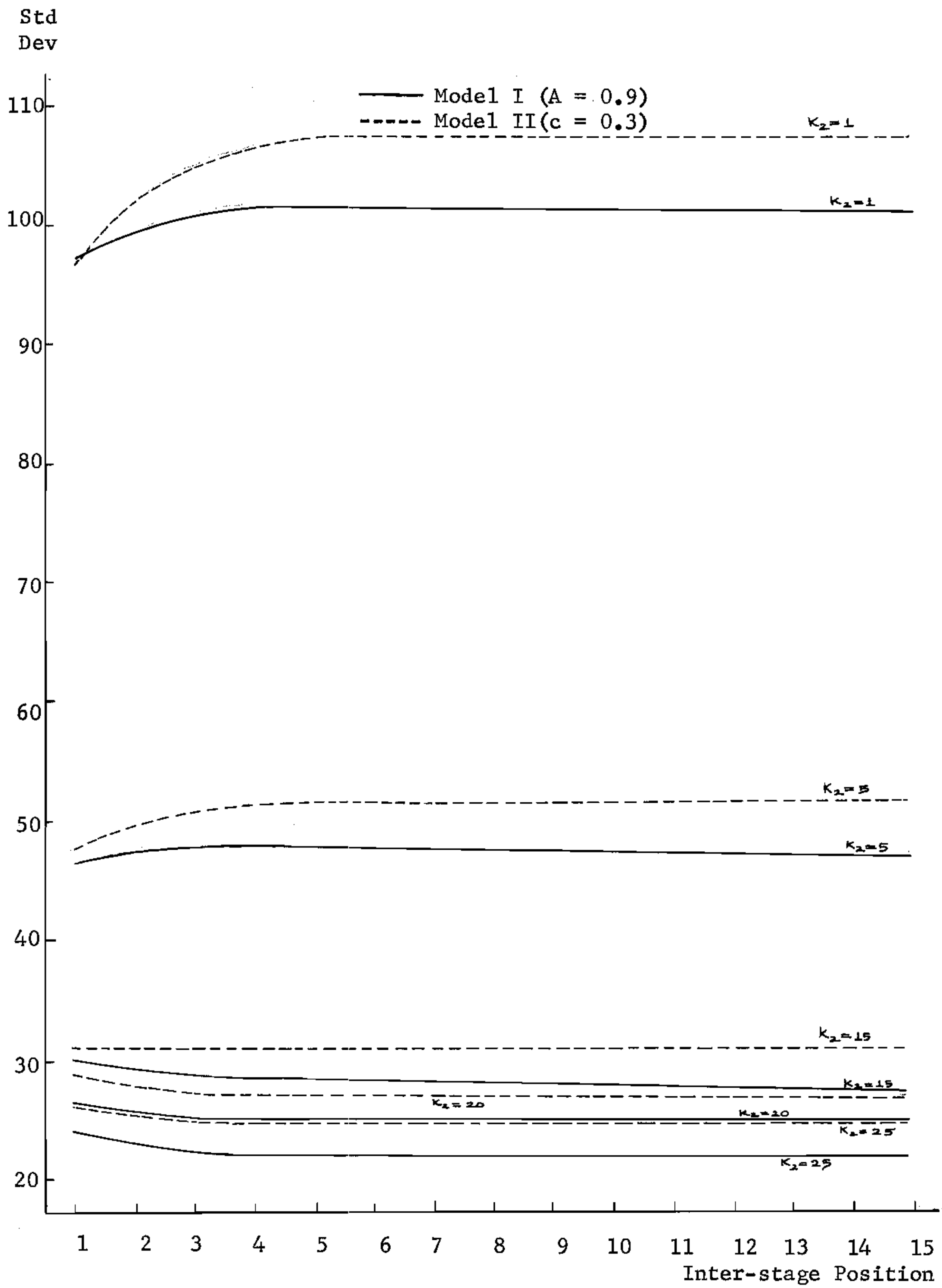

Figure 9. Standard Deviation of the Inter-departure Time Distribution Shown Against the Inter-stage Position - Erlangian Arrivals ($K_1 = 10$)

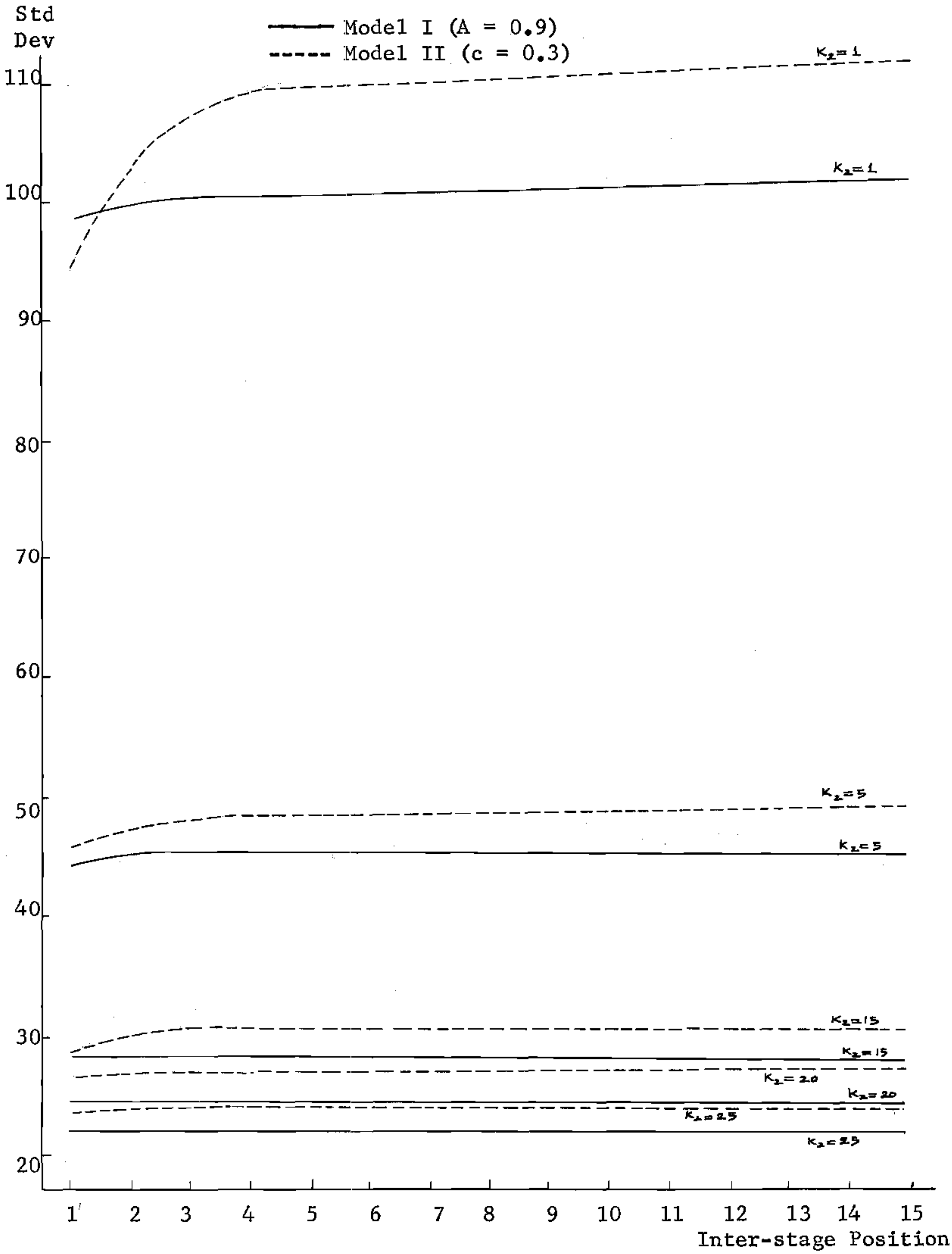

Figure 10. Standard Deviation of the Inter-departure Time Distribution Shown Against the Inter-stage Position - Erlangian Arrivals (K_1 = 20)

parameters in Table 2. The corresponding tables at stages five, ten, and 15 are shown in Appendix B. The following observations are made from the figures shown:

1. The departure distribution from any stage will in general have a greater variability than the service distribution at that stage. This is a consequence of the variation in the mean service rate and of a possible idle system.

2. When the arrival distribution to the first stage (with Erlang parameter K_1) has a greater variability than the service time distribution at each stage (parameter K_2) the standard deviation of the output distribution decreases, within a limit, from one stage to the next stage. After this limit the standard deviation tends to remain constant. Further the slope of the curve tends to zero, the slope decreasing from one stage to the subsequent stage.

3. When K_1 is greater than or equal to K_2 (when the arrivals to the first stage are more regular than, or as regular as, the services at each stage) the standard deviation of the output distribution increases with the inter-stage position (with the slope of the curve decreasing) up to a certain extent after which it approaches a constant value.

The effect of changing the parameters in model I and model II can be clearly seen from Table 2. The variability of the output distribution at any stage increases with parameter B in model I and with c in model II.

Another interesting characteristic is observed when the output from a stage has a variability not greater than that of an exponential distribution. In such a case the output can be approximated to an Erlang distribution with parameter K_3. K_3 is estimated using the relationship

Table 2. Standard Deviation of the Inter-departure Time Distribution at the First Stage of a Sequential Waiting Line System.

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	100.82	97.18	97.28	98.85					
.8	101.42	94.59	93.53	95.69	.3	109.17	97.93	96.60	94.08
.7	103.88	95.91	95.68	98.64	.5	109.85	98.74	98.37	97.31
.6	106.36	95.35	95.26	96.68	1.0	118.60	101.95	100.54	100.09
.5	109.22	99.00	96.27	99.49					
A	**AB1**	**BB1**	**FB1**	**DB1**	**c**	**AB2**	**BB2**	**FB2**	**DB2**
.9	58.16	47.32	46.54	44.01					
.8	61.46	49.61	46.50	45.16	.3	73.37	50.15	47.17	45.87
.7	70.04	49.81	48.39	44.56	.5	79.54	52.27	50.35	47.21
.6	76.41	51.59	47.27	47.67	1.0	85.35	55.89	51.68	48.67
.5	81.06	52.86	50.77	49.11					
A	**AC1**	**BC1**	**FC1**	**DC1**	**c**	**AC2**	**BC2**	**FC2**	**DC2**
.9	49.06	29.54	28.64	28.59	.3	73.53	33.55	31.00	28.92
.8	59.12	31.59	30.78	29.43	.3	73.53	33.55	31.00	28.92
.7	69.20	32.25	30.56	29.18	.5	76.35	35.78	31.97	31.18
.6	68.35	33.82	32.03	30.79	1.0	84.41	39.59	35.79	32.34
.5	75.65	36.87	33.61	31.62					
A	**AD1**	**BD1**	**FD1**	**DD1**		**AD2**	**BD2**	**FD2**	**DD2**
.9	47.14	26.54	26.50	24.73					
.8	58.46	28.43	27.55	24.99	.3	62.64	30.13	28.83	26.96
.7	62.23	29.98	27.77	26.11	.5	77.10	33.18	30.22	27.28
.6	71.41	32.66	29.71	26.87	1.0	76.52	37.23	32.30	29.35
.5	71.90	33.69	30.56	27.73					
A	**AE1**	**BE1**	**FE1**	**DE1**		**AE2**	**BE2**	**FE2**	**DE2**
.9	48.22	23.54	23.62	22.73					
.8	57.35	27.19	25.58	23.27	.3	59.08	27.57	26.27	23.87
.7	59.63	29.36	26.03	24.47	.5	71.55	31.83	27.46	25.34
.6	62.94	30.14	266655	25.28	1.0	79.24	34.98	30.08	27.11
.5	67.52	31.57	28.15	25.32					

$$K_3 = \frac{\text{Mean of the output distribution}}{\text{Variance of the output distribution}}^2 \qquad (27)$$

If the value of K_3 thus estimated turns out to be a decimal number, the next larger integer is taken to be the value of K_3.

The output distributions are compared with the corresponding Erlang distribution using the X^2 goodness of fit test. It is found that the two distributions do not differ from each other significantly. The difference is the least when the estimated value of K_3 is large.

Average Waiting Times at Various Stages

In the study of a tandem queueing system, the average waiting time of a unit at each stage (the mean of the waiting time distribution at that stage) constitutes a good measure of effectiveness. In many cases the cost of waiting may be assumed to be directly proportional to the mean waiting time.

The expected waiting times at the first stage for various levels of service are shown in Table 3, and are plotted for a typical set of parameters in Figure 11. It is interesting to note that the waiting times resulting from the use of model I and model II do not differ significantly when the parameter K_2 in the service time distribution is greater than or equal to five. This comparison is made possible by selecting the same scale for B and C along the absissa of the figure. The expected waiting times at stages five, 10, and 15 are tabulated in Appendix B.

The mean waiting times are also shown as dependent variables against the inter-stage position. These are shown in Appendix B. An example of the above is shown in Figure 12. Here again the curves are

Table 3. Mean Waiting Times at the First Stage of a Series Queueing System

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	823.05	464.99	439.50	430.50					
.8	313.13	242.17	219.45	212.20	.3	170.98	155.22	134.68	134.31
.7	248.57	173.31	160.76	156.20	.5	131.37	111.18	107.80	105.21
.6	194.43	134.44	126.58	125.15	1.0	88.53	80.41	80.14	78.52
.5	134.50	119.68	101.16	100.37					

A	AB1	BB1	FB1	DB1	c	AB2	BB2	FB2	DB2
.9	450.28	173.32	166.92	127.64					
.8	229.43	118.18	96.22	71.18	.3	130.92	83.58	80.67	64.65
.7	151.56	86.23	70.26	69.80	.5	91.71	64.21	58.29	55.60
.6	109.83	69.21	62.61	60.19	1.0	64.35	48.34	48.03	44.05
.5	96.91	62.83	61.00	56.52					

A	AC1	BC1	FC1	DC1	c	AC2	BC2	FC2	DC2
.9	349.00	142.59	117.11	87.36					
.8	208.23	90.28	76.31	64.83	.3	121.08	67.43	59.99	57.51
.7	132.28	77.47	63.22	60.61	.5	84.87	56.88	52.69	46.26
.6	97.46	63.27	56.28	49.88	1.0	57.29	45.61	44.20	39.69
.5	86.90	52.93	47.88	46.72					

A	AD1	BD1	FD1	DD1	c	AD2	BD2	FD2	DC3
.9	335.92	141.05	99.38	82.16					
.8	198.95	89.81	75.41	59.91	.3	122.12	67.21	58.00	51.77
.7	130.08	73.43	58.47	54.90	.5	84.67	53.68	50.99	45.23
.6	96.21	61.29	53.18	46.50	1.0	58.21	43.81	43.12	43.11
.5	82.35	53.88	47.10	44.63					

A	AE1	BE1	FE1	DE1	c	AE2	BE2	FE2	DC4
.9	291.92	140.63	91.12	79.17					
.8	167.56	90.36	74.82	63.11	.3	121.99	66.87	56.39	51.81
.7	123.53	67.68	57.21	53.02	.5	83.37	55.01	48.84	46.85
.6	95.53	57.13	50.82	46.21	1.0	58.04	45.39	43.16	42.30
.5	80.75	54.49	49.15	43.85					

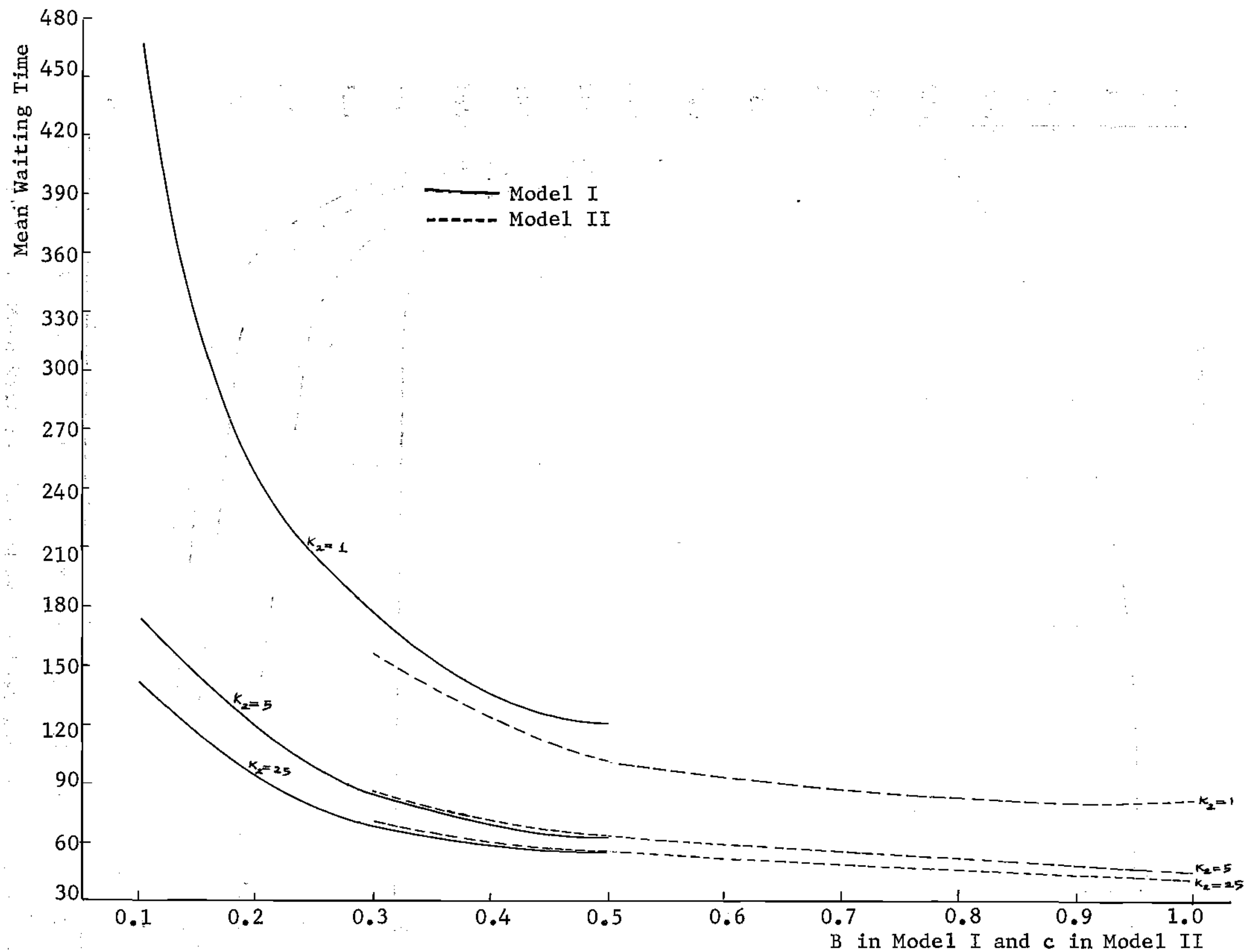

Figure 11. Mean Waiting Times at the First Stage Shown Against Various Levels of Service - Erlangian Arrivals (K_1 = 5)

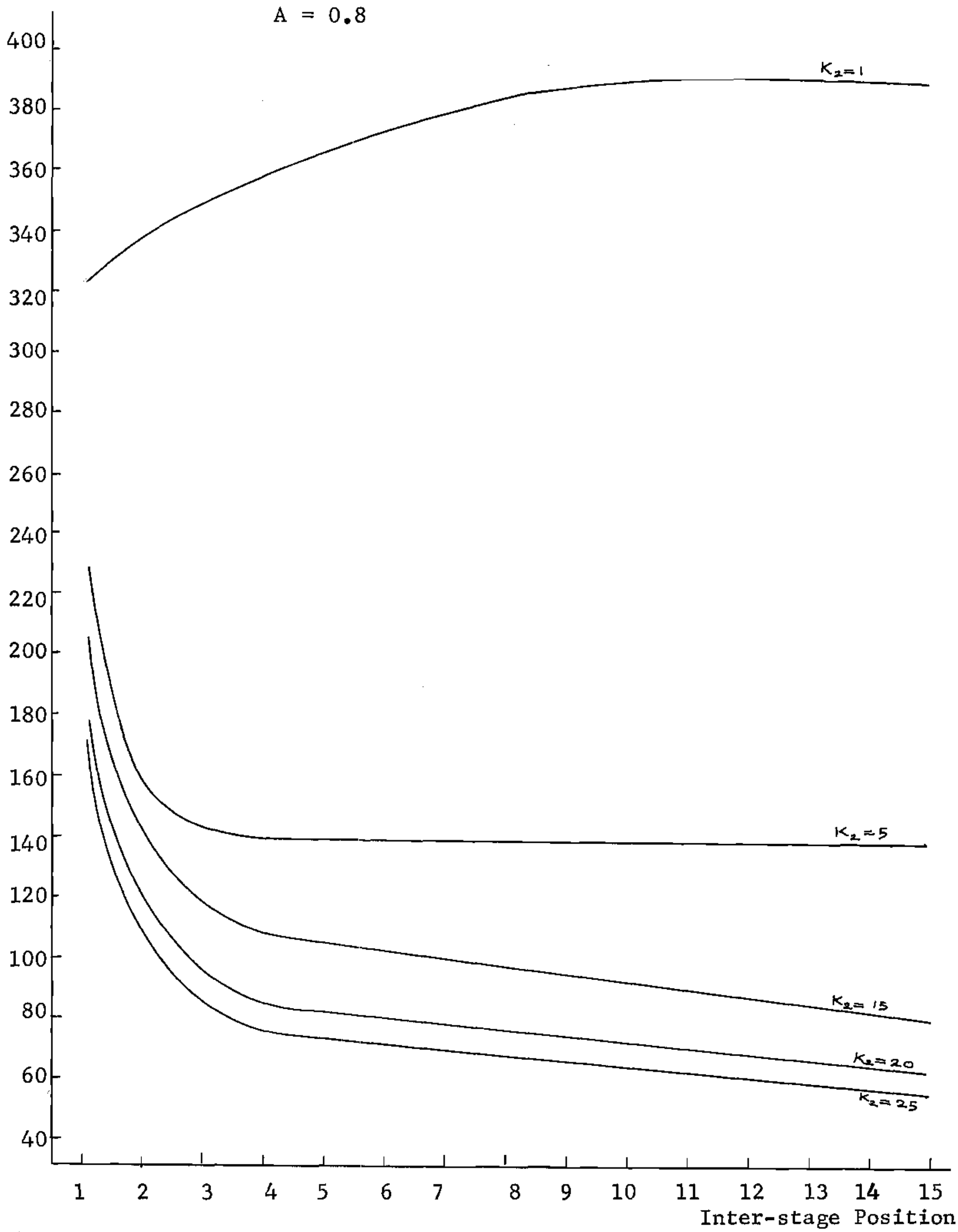

Figure 12. Mean Waiting Times Shown Against Inter-Stage Position - Model I - Poisson Arrivals (K_1 = 1)

shown to be continuous for convenience. It is found that the observations made on the standard deviation of the inter-departure time distribution, with regard to the nature of its behavior with the position of the inter-stage, are also applicable for the mean waiting times. It is found that the mean waiting times at the first stage, as the value of A in model I approaches unity, approximates that obtained when there is no state dependency relationship with $\rho_n = \rho = \frac{\lambda}{\mu} = A$.

Distribution of the State Probabilities

Generally the word "state probability" implies that the state of the system, n, (the number in queue plus the number in the service facility) and its associated steady-state probability, π_n , are being considered. However, for purposes of this research, n_q, the number in the queue, (also called the state of the queue) is taken in establishing the state probability distributions. In a queueing problem, where the objective of the designer is to establish the queue capacity for a service stage or between any two stages, the probabilities in state n_q are of interest. The mean of the state probability distribution is simply the average queue length (L_q). From the theory of queueing, the expected length of the queue is the product of the mean waiting time and the effective arrival rate. Effective arrival rate measures only those units which join the system. Since in this study, balking, reneging, and other complications are not allowed, all the incoming arrivals join the system and stay in the system till they get served.

Thus,

$$L_q = (\text{Mean waiting time})\ (\lambda) = \frac{\text{Mean waiting time}}{100} \tag{28}$$

Hence the curves showing the variation of the mean queue length with the position of the interstage are similar to those given in Figure 12, and the conclusions drawn regarding the mean waiting times also hold for the means of the various state probability distributions. The state probability distributions at the first stage for various input parameters are given in Appendix B. One such distribution for a particular combination of parameters is shown in Figure 13. As it is obvious, only an integer value of n_q is to be taken to obtain the corresponding probability from the figure.

From a state probability distribution one can readily obtain the following:

1. The fraction of time a unit gets served without delay. This is simply the probability of having none in the queue. It is also the sum of the probability of an idle system and the probability of having one in the system.

2. The probability that the number in queue is less than or equal to n_q. From this quantity the fraction of time the number exceeds n_q can be computed (by subtracting the former from one).

3. The probability in state n_q, which is obtained from the following relationship:

$$p(n_q) \quad = \quad P(N_q \leq n_q) \quad - \quad P(N_q \leq n_{q-1}) \tag{29}$$

4. The maximum queue length. This is the value of n_q for which $P\ (N_q > n_q) = 0$.

The average utilization of the service facility at stages one,

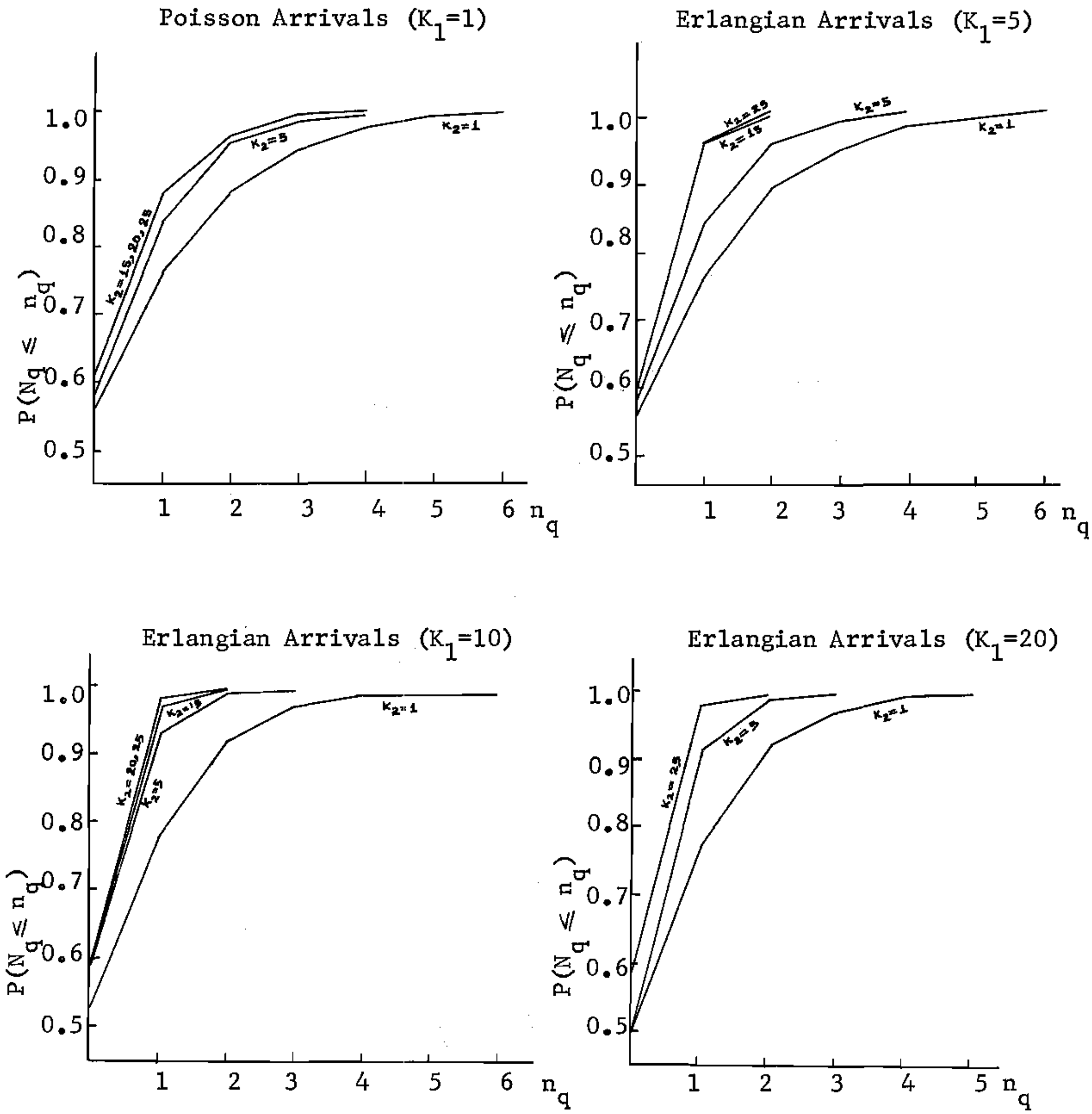

Figure 13. The State Probability Distribution at the First Stage for Various K_1 and K_2 - Model II (c=1.0)

five, ten, and 15 are given in Appendix B from which the probability of an idle system may be calculated. It is noticed that when $K_2 \leq K_1$ the average utilization tends to decrease from one stage to the next stage. This is an agreement with the conclusions drawn regarding the variability of the output distribution at each stage. When $K_2 \leq K_1$, the mean queue length tends to increase from one stage to the next stage which will cause the corresponding service facility to work faster. When the service facility works faster it is bound to finish the work early and be idle for a longer time. Longer idle time implies lower utilization which occurs when the value of B or c in the service distribution is increased. It is also seen that when the arrivals or service times at any stage are more regular, a smaller number of units wait for service ($P(N_q > n_q)$ is smaller), and this results in a higher utilization of the facility. Following the conventions of the experimental coding developed in Chapter III, any run XY (when the Erlang parameter K_1 of X is less than the parameter K_2 of Y) results in a lower mean queuellength than the run YX.

Figure 14 is given to show the behavior of the state probability distribution at the first stage when the various parameters are kept constant. Similar behavior is observed at other stages of the series queue as well. Comparison of Figures 13 and 14a leads to the conclusion that the state probability distribution at any stage is less sensitive to the parameter K_1 in the arrival distribution than to the parameter K_2 in the service distribution. Figure 14b shows the effect of varying the value of c in model II when both K_1 and K_2 have a relatively high variability, while Figure 14c shows the effect when the arrivals and services

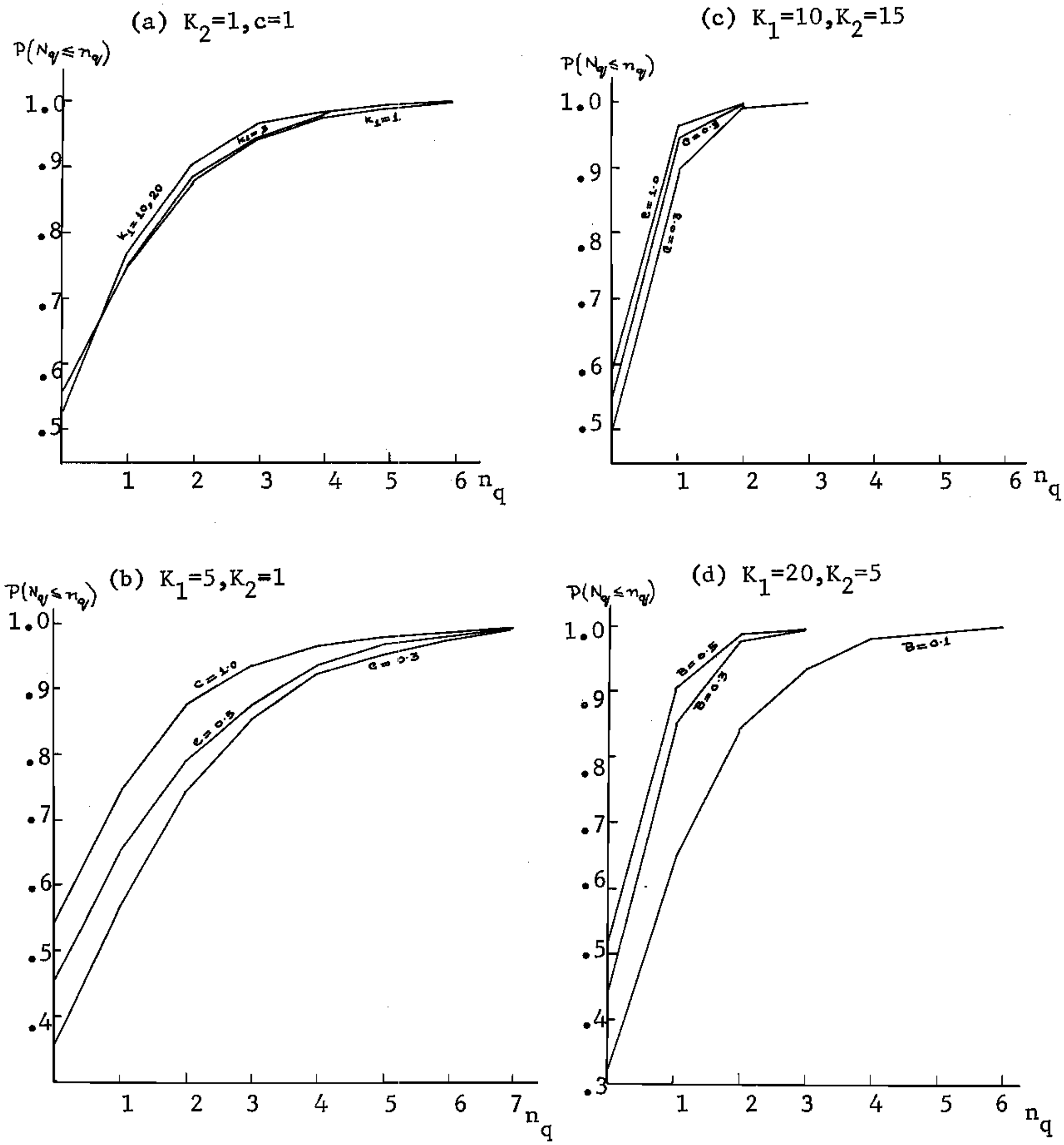

Figure 14. The State Probability Distribution at the First Stage when the Values of Various Parameters are Held Constant

are more regular. Figure 14d shows the state probability distributions for different values of parameter B in model I, when $K_1 = 20$ and $K_2 = 5$.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The following conclusions are drawn from this research:

1. The assumption that the mean service time is a function of the number of units awaiting service at the moment the service of any unit starts, results in a considerably higher probability of waiting for service than a case when the service rate is dependent upon the instantaneous state of the system.

2. When the parameter A in model I tends to unity, the results obtained from the investigation on a state dependent queueing system, where the arrival rate is equal to the base mean service rate, approaches those of a system where the arrivals and services are independent of each other with the coefficient of utilization equal to A.

3. When the services at each stage in a tandem queueing system follow quite regular Erlangian times, the output from any stage of a state dependent system can be reasonably approximated by an Erlang distribution. Eventually this leads to the conclusion that for a particular set of parameters Erlangian arrivals and services result in Erlangian output with the same parameter K_1 as in the input distribution.

4. The output of a state dependent queueing system, where the arrivals follow the Poisson distribution and the services follow an exponential distribution, is not Poisson.

There seems to be a lot of scope for a further investigation of the behavior of tandem queues. As it was observed a mathematical approach to such a problem tends to be intractable, and a simulation approach seems to be inevitable. Further studies on tandem queues can be made by allowing only a limited queue between stages to study the effect of blocking at each stage. The steady state joint probability of having a specific number of units at different stages may be studied to compute the remaining work content of the series queueing system.

APPENDIX A

RESUMÉ OF GPSS

GPSS (General Purpose Systems Simulator) was developed by Gordon at IBM for use on the IBM 7090 computer. It isaa simulation language designed specifically for modeling queueing systems. The user constructs a logical model of the system using a block diagram consisting of specific block types, in which each block type represents some basic system action. Each block has a designated block time (action time) that indicates the number of time units required for action represented by the block. The action times may vary over a range of values in a random or non-random manner. Transactions are basic units that move through the system. Parameters are integral positive quantities that can be attached to a transaction. Facilities are items of equipment that can handle one transaction at a time. Stores are items of equipment that can handle many transactions simultaneously.

In the computer program which is composed from a GPSS block diagram, one statement corresponds to one block, and one statement is punched on one card. Each card identifies the block it represents by a number (block number) and the block type. Each punched card is divided into eleven fields. The number of the block is punched in the first field. The block type is punched in the next field. The number of the block to which a departing unit attempts to enter is specified in the seventh field. The remaining fields are used to represent the action times, logic of the flow of transactions, etc.

Indirect specification isaa technique used in GPSS to increase the ability of the program to represent systems. It can also substantially decrease the size of the block diagram and/or increase the ease of programming. An asterisk associated with a parameter number (*5, say) constitutes indirect specification in GPSS. To cite an example, if the facility number in a HOLD block is specified as *5, the program takes the number in parameter field five of the transaction entering the HOLD block as the number of the facility.

The following are the block types used in the simulation model. A brief description of the block types is also given.

ORIGINATE creates transactions at time intervals specified in the block and enters them into the system being simulated.

ASSIGN assigns a value to one of the eight parameters of a transaction. This block can either add to, subtract from or replace the previous value assigned to the parameter field of the transaction entering the block.

QUEUE maintains various statistics about queues.

HOLD allows a transaction to engage the facility for as long as the transaction remains in the block.

TABULATE records the statistics specified in the block resulting in a print-out in histogram form.

TERMINATE removes transactions from the system the instant they enter the block.

COMPARE controls the flow of transactions by comparing two different values specified in the block.

```
       JOB
*      SIMULATION OF A SEQUENTIAL QUEUEING SYSTEM WITH
*      STATE DEPENDENT MEAN SERVICE RATES
*      NUMBER OF STAGES OF THE QUEUE ( 15
*      DEFINE THE ARRIVAL DISTRIBUTION
 1     FUNCTION     RN1   C24            EXPONENTIAL MEAN(100
 0     0      .1    10.40 .2    22.20 .3    35.50 .4    50.90 .5     69
 .6    91.50 .7     120.00.75   138.00.8    160.00.84   183.00.88    212.00
 .9    230.00.92    252.00.94   281.00.95   299.00.96   320.00.97    350.00
 .98   390.00.99    460.00.995  530.00.998  620.00.999  700.00.9997  800.00
*      DEFINE THE SERVICE TIME DISTRIBUTION
 2     FUNCTION     RN1   C24            EXPONENTIAL MEAN(100
 0     0      .1    10.40 .2    22.20 .3    35.50 .4    50.90 .5     69
 .6    91.50 .7     120.00.75   138.00.8    160.00.84   183.00.88    212.00
 .9    230.00.92    252.00.94   281.00.95   299.00.96   320.00.97    350.00
 .98   390.00.99    460.00.995  530.00.998  620.00.999  700.00.9997  800.00
*      DEFINE THE STATE DEPENDENCY RELATIONSHIP
 3     FUNCTION     Q*2   D21            C=1
 0     1      1     .500  2     .333  3     .250  4     .200  5      .167
 6     .143   7     .125  8     .111  9     .100  10    .091  11     .083
 12    .077   13    .0714 14    .0667 15    .0625 16    .0588 17     .0555
 18    .0526 19     .050  20    .0476
 1     ORIGINATE                         2            1     FN1
 2     ASSIGN       1     FN2            3
 3     ASSIGN       2     K1             4
 4     QUEUE        *2                   5
 5     HOLD         *2                   6            *1    FN3
 6     TABULATE     *2                   7
 7     ASSIGN       1     FN2            8
 8     ASSIGN       2     K2             9
 9     QUEUE        *2                   10
 10    HOLD         *2                   11           *1    FN3
 11    TABULATE     *2                   12
 12    ASSIGN       1     FN2            13
 13    ASSIGN       2     K3             14
```

14	QUEUE	*2		15		
15	HOLD	*2		16	*1	FN3
16	TABULATE	*2		17		
17	ASSIGN	1	FN2	18		
18	ASSIGN	2	K4	19		
19	QUEUE	*2		20		
20	HOLD	*2		21	*1	FN3
21	TABULATE	*2		22		
22	ASSIGN	1	FN2	23		
23	ASSIGN	2	K5	24		
24	QUEUE	*2		25		
25	HOLD	*2		26	*1	FN3
26	TABULATE	*2		27		
27	ASSIGN	1	FN2	28		
28	ASSIGN	2	K6	29		
29	QUEUE	*2		30		
30	HOLD	*2		31	*1	FN3
31	TABULATE	*2		32		
32	ASSIGN	1	FN2	33		
33	ASSIGN	2	K7	34		
34	QUEUE	*2		35		
35	HOLD	*2		36	*1	FN3
36	TABULATE	*2		37		
37	ASSIGN	1	FN2	38		
38	ASSIGN	2	K8	39		
39	QUEUE	*2		40		
40	HOLD	*2		41	*1	FN3
41	TABULATE	*2		42		
42	ASSIGN	1	FN2	43		
43	ASSIGN	2	K9	44		
44	QUEUE	*2		45		
45	HOLD	*2		46	*1	FN3
46	TABULATE	*2		47		
47	ASSIGN	1	FN2	48		

```
 48    ASSIGN       2     K10               49
 49    QUEUE        *2                      50
 50    HOLD         *2                      51        *1     FN3
 51    TABULATE     *2                      52
 52    ASSIGN       1     FN2               53
 53    ASSIGN       2     K11               54
 54    QUEUE        *2                      55
 55    HOLD         *2                      56        *1     FN3
 56    TABULATE     *2                      57
 57    ASSIGN       1     FN2               58
 58    ASSIGN       2     K12               59
 59    QUEUE        *2                      60
 60    HOLD         *2                      61        *1     FN3
 61    TABULATE     *2                      62
 62    ASSIGN       1     FN2               63
 63    ASSIGN       2     K13               64
 64    QUEUE        *2                      65
 65    HOLD         *2                      66        *1     FN3
 66    TABULATE     *2                      67
 67    ASSIGN       1     FN2               68
 68    ASSIGN       2     K14               69
 69    QUEUE        *2                      70
 70    HOLD         *2                      71        *1     FN3
 71    TABULATE     *2                      72
 72    ASSIGN       1     FN2               73
 73    ASSIGN       2     K15               74
 74    QUEUE        *2                      75
 75    HOLD         *2                      76        *1     FN3
 76    TABULATE     *2                      77
 77    TERMINATE    R
*      THE FOLLOWING TABLE CARDS GIVE THE INTERDEPARTURE TIME
*      DISTRIBUTIONS.
 1     TABLE        IA    0     10    100
```

```
 2      TABLE         IA     0       10      100
 3      TABLE         IA     0       10      100
 4      TABLE         IA     0       10      100
 5      TABLE         IA     0       10      100
 6      TABLE         IA     0       10      100
 7      TABLE         IA     0       10      100
 8      TABLE         IA     0       10      100
 9      TABLE         IA     0       10      100
 10     TABLE         IA     0       10      100
 11     TABLE         IA     0       10      100
 12     TABLE         IA     0       10      100
 13     TABLE         IA     0       10      100
 14     TABLE         IA     0       10      100
 15     TABLE         IA     0       10      100
*       THE FOLLOWING QTABLE CARDS GIVE THE WAITING TIME DISTRIBUTIONS
*       OF QUEUES 1 THRU 15
 16     QTABLE        1      0       10      100
 17     QTABLE        2      0       10      100
 18     QTABLE        3      0       10      100
 19     QTABLE        4      0       10      100
 20     QTABLE        5      0       10      100
 21     QTABLE        6      0       10      100
 22     QTABLE        7      0       10      100
 23     QTABLE        8      0       10      100
 24     QTABLE        9      0       10      100
 25     QTABLE        10     0       10      100
 26     QTABLE        11     0       10      100
 27     QTABLE        12     0       10      100
 28     QTABLE        13     0       10      100
 29     QTABLE        14     0       10      100
 30     QTABLE        15     0       10      100
*       PROGRAM FOR COMPUTING AND TABULATING THE STATE PROBABILITY
*       DISTRIBUTIONS
```

```
 90    ORIGINATE                                    91           50
 1     VARIABLE     P80K30
 91    ASSIGN       80     K1                       92
 92    ASSIGN       7      V1                       93
 93    TABULATE     *7                      BOTH    94     91
 94    COMPARE      P8     GE     K15               95
 95    TERMINATE    R
 31    TABLE        Q1     0      1         70
 32    TABLE        Q2     0      1         70
 33    TABLE        Q3     0      1         70
 34    TABLE        Q4     0      1         70
 35    TABLE        Q5     0      1         70
 36    TABLE        Q6     0      1         70
 37    TABLE        Q7     0      1         70
 38    TABLE        Q8     0      1         70
 39    TABLE        Q9     0      1         70
 40    TABLE        Q10    0      1         70
 41    TABLE        Q11    0      1         70
 42    TABLE        Q12    0      1         70
 43    TABLE        Q13    0      1         70
 44    TABLE        Q14    0      1         70
 45    TABLE        Q15    0      1         70
*      STABILIZE THE SYSTEM
       START        1500   NP
       RESET
*      SAMPLE SIZE
       START        18000
       END
```

APPENDIX B

Table 4. Standard Deviation of the Inter-departure Time Distribution at the Fifth Stage of a Sequential Waiting Line System

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	101.69	101.04	99.66	99.74					
.8	105.80	100.48	101.01	99.95	.3	113.79	110.60	106.11	111.76
.7	106.69	105.17	107.43	104.72	.5	118.46	114.24	114.01	115.08
.6	111.39	109.91	107.85	110.29	1.0	134.09	128.00	124.88	129.42
.5	117.58	115.51	112.18	112.24					

A	AB1	BB1	FB1	DB1	c	AB2	BB2	FB2	DB2
.9	50.21	47.36	47.75	46.65					
.8	53.24	49.72	30.91	47.24	.3	62.16	51.41	51.58	48.85
.7	56.75	52.37	52.82	49.15	.5	66.00	56.37	55.29	52.98
.6	65.34	55.54	55.08	54.51	1.0	74.39	63.71	61.97	59.13
.5	67.18	56.59	55.89	54.58					

A	AC1	BC1	FC1	DC1	c	AC2	BC2	FC2	DC2
.9	36.80	28.97	28.71	29.02					
.8	42.73	30.13	30.53	30.90	.3	51.12	32.89	30.95	30.96
.7	48.85	31.14	31.18	30.26	.5	59.92	35.07	32.89	32.48
.6	51.77	32.95	33.06	33.20	1.0	64.25	38.70	37.43	36.93
.5	56.64	34.11	35.08	32.90					

A	AD1	BD1	FD1	DD1	c	AD2	BD2	FD2	DD2
.9	34.97	24.91	25.52	26.05					
.8	39.88	26.91	25.72	26.39	.3	43.61	28.26	28.04	26.78
.7	44.25	28.07	27.11	26.82	.5	56.85	29.82	29.01	28.89
.6	49.54	29.47	28.74	28.74	1.0	58.94	34.84	32.40	32.19
.5	50.90	30.20	29.70	28.63					

A	AE1	BE1	FE1	DE1	c	AE2	BE2	FE2	DE2
.9	34.82	22.19	22.48	23.27					
.8	40.21	23.30	23.49	23.56	.3	38.27	25.44	24.47	24.34
.7	42.74	25.99	24.41	24.77	.5	49.49	27.49	26.40	25.76
.6	43.83	26.66	25.09	24.94	1.0	56.46	30.86	29.62	28.85
.5	46.44	27.81	26.61	25.31					

Table 5. Standard Deviation of the Inter-departure Time Distribution at the Tenth Stage of a Sequential Waiting Line System

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	100.66	99.79	100.35	102.60					
.8	104.79	102.12	107.39	100.43	.3	112.92	107.39	109.06	114.43
.7	110.89	106.54	106.66	107.23	.5	116.96	115.45	115.39	115.65
.6	115.53	107.45	109.49	113.93	1.0	139.35	137.05	136.81	140.11
.5	116.85	113.39	116.91	114.25					

A	AB1	BB1	FB1	DB1	c	AB2	BB2	FB2	DB2
.9	49.62	48.10	48.01	46.99					
.8	51.38	51.20	49.72	48.53	.3	58.25	54.66	52.17	47.36
.7	54.27	52.38	51.34	49.55	.5	65.57	55.86	57.31	52.83
.6	63.76	54.32	53.65	53.42	1.0	71.14	65.91	63.20	59.31
.5	65.05	55.65	56.18	56.82					

A	AC1	BC1	FC1	DC1	c	AC2	BC2	FC2	DC2
.9	32.48	28.57	28.02	29.11					
.8	37.31	30.36	30.14	31.36	.3	42.42	31.43	31.05	31.84
.7	43.27	30.56	31.57	30.94	.5	46.92	34.05	33.36	33.82
.6	44.45	33.65	32.84	33.36	1.0	53.46	38.66	37.81	37.26
.5	49.58	35.12	34.54	33.54					

A	AD1	BD1	FD1	DD1	c	AD2	BD2	FD2	DD2
.9	31.23	24.18	25.16	24.94					
.8	33.82	26.59	26.72	26.67	.3	35.72	28.10	27.95	27.88
.7	40.06	26.38	27.78	27.19	.5	48.19	29.51	29.19	28.80
.6	40.53	28.99	27.86	28.75	1.0	44.31	34.20	31.64	31.62
.5	45.53	30.28	29.03	29.95					

A	AE1	BE1	FE1	DE1	c	AE2	BE2	FE2	DE2
.9	30.53	22.45	22.85	22.50					
.8	36.51	23.21	23.60	23.36	.3	32.27	24.49	24.46	24.45
.7	37.71	24.74	24.47	24.54	.5	40.04	27.13	26.05	25.65
.6	37.46	26.27	24.14	24.58	1.0	46.77	30.09	28.81	28.93
.5	37.98	26.38	26.48	25.58					

Table 6. Standard Deviation of the Inter-departure Time Distribution at the Fifteenth Stage of a Sequential Waiting Line System

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	102.38	102.78	104.06	104.99					
.8	105.16	103.08	104.96	104.11	.3	114.26	107.28	106.81	110.31
.7	104.81	108.54	106.97	102.39	.5	118.17	120.96	113.77	118.62
.6	112.22	112.18	110.19	108.70	1.0	142.52	138.27	142.86	140.49
.5	117.17	117.12	116.59	121.77					
A	AB1	BB1	FB1	DB1	c	AB2	BB2	FB2	DB2
.9	48.56	47.52	48.08	45.65					
.8	51.30	50.37	50.20	48.07	.3	58.70	53.62	52.45	49.71
.7	52.87	52.65	53.02	49.32	.5	63.23	57.58	56.65	53.93
.6	60.97	55.84	53.84	53.66	1.0	70.88	64.86	63.44	50.23
.5	64.23	54.64	56.42	57.38					
A	AC1	BC1	FC1	DC1	c	AC2	BC2	FC2	DC2
.9	31.06	28.41	28.47	28.93					
.8	37.73	29.72	30.39	30.41	.3	39.60	31.75	31.27	30.46
.7	41.10	30.75	32.30	30.87	.5	44.08	34.19	33.05	33.48
.6	41.18	32.77	33.56	32.85	1.0	50.13	38.61	37.66	37.14
.5	45.66	34.84	34.61	33.85					
A	AD1	BD1	FD1	DD1	c	AD2	BD2	FD2	DD2
.9	29.77	24.21	25.16	25.48					
.8	30.55	26.51	26.07	26.48	.3	33.97	27.99	27.44	26.81
.7	36.86	27.51	26.87	27.12	.5	44.26	29.04	28.46	28.60
.6	37.56	28.87	27.93	28.19	1.0	40.55	34.54	32.22	32.03
.5	42.46	29.38	229.65	29.38					
A	AE1	BE1	FE1	DE1	c	AE2	BE2	FE2	DE2
.9	28.00	21.59	22.33	22.76					
.8	33.00	22.99	23.49	23.55	.3	28.65	24.88	24.01	24.07
.7	33.93	24.95	23.80	24.29	.5	36.10	26.32	25.63	26.06
.6	33.50	26.06	24.65	25.36	1.0	41.10	29.65	28.98	29.12
.5	33.72	26.29	25.33	26.07					

Table 7. Mean Waiting Times of the Fifth Stage of a Service Queueing System

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	1227.26	627.21	634.77	644.60					
.8	429.50	383.16	342.62	407.90	.3	190.20	191.35	190.18	177.03
.7	231.86	227.72	228.18	227.80	.5	141.43	130.07	135.98	144.31
.6	191.83	174.65	190.72	191.30	1.0	98.69	95.13	91.62	98.06
.5	159.10	144.42	151.52	143.40					

A	AB1	BB1	FB1	DB1	c	AB2	BB2	FB2	DB2
.9	198.37	199.70	208.15	149.20					
.8	142.98	125.23	115.76	98.27	.3	94.19	90.99	83.77	76.12
.7	96.73	84.25	83.95	81.21	.5	71.67	68.37	68.49	62.23
.6	86.89	72.10	73.84	75.45	1.0	55.55	51.02	53.01	50.23
.5	76.29	70.05	65.21	66.34					

A	AC1	BC1	FC1	DC1	c	AC2	BC2	FC2	DC2
.9	131.77	108.05	105.61	103.42					
.8	99.35	80.88	71.94	65.69	.3	78.72	64.49	57.60	55.97
.7	72.56	70.53	62.80	63.21	.5	60.58	53.31	54.15	51.75
.6	60.18	54.98	54.40	51.53	1.0	46.77	43.86	43.04	43.79
.5	60.51	53.01	49.40	51.39					

A	AD1	BD1	FD1	DD1	c	AD2	BD2	FD2	DD2
.9	127.11	117.58	83.28	80.30					
.8	89.39	78.56	68.38	62.70	.3	81.61	61.54	57.75	56.36
.7	73.46	63.65	58.99	58.90	.5	58.21	51.85	48.20	49.98
.6	61.95	53.41	54.85	48.11	1.0	47.62	42.72	45.04	42.72
.5	54.52	51.31	49.20	49.52					

A	AE1	BE1	FE1	DE1	c	AE2	BE2	FE2	DE2
.9	85.60	98.83	70.82	78.46					
.8	69.83	76.13	70.85	63.11	.3	77.87	62.68	53.18	53.19
.7	66.97	61.39	55.77	54.10	.5	57.65	52.15	47.00	47.88
.6	66.52	53.51	49.90	51.23	1.0	47.34	45.44	42.55	441.20
.5	60.98	51.76	45.96	45.62					

Table 8. Mean Waiting Times at the Tenth Stage of a Series Queueing System

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	961.06	820.52	619.80	584.10					
.8	403.17	389.17	431.22	431.20	.3	177.58	184.95	191.48	202.13
.7	246.96	252.98	255.44	255.80	.5	145.06	138.24	138.46	132.16
.6	192.21	186.53	186.09	178.30	1.0	96.88	98.28	95.09	95.83
.5	153.56	148.01	148.13	151.53					

A	AB1	BB1	FB1	DB1	c	AB2	BB2	FB2	DB2
.9	240.59	184.18	181.64	168.30					
.8	152.46	111.62	106.45	95.41	.3	92.15	83.37	92.46	78.02
.7	96.57	89.30	84.85	82.10	.5	72.03	67.82	68.71	64.48
.6	80.64	73.09	71.61	75.57	1.0	54.13	53.16	55.42	50.35
.5	73.33	71.16	70.17	69.10					

A	AC1	BC1	FC1	DC1	c	AC2	BC2	FC2	DC2
.9	118.39	117.37	112.58	100.85					
.8	86.89	77.80	70.39	66.64	.3	66.77	64.21	63.77	59.53
.7	63.52	74.39	61.40	61.73	.5	76.06	51.61	51.64	51.11
.6	59.04	58.43	56.43	52.23	1.0	43.77	45.54	44.60	43.38
.5	55.81	50.07	52.04	50.35					

A	AD1	BD1	FD1	DD1	c	AD2	BD2	FD2	DD2
.9	109.10	107.93	84.93	91.02					
.8	73.97	78.05	66.90	62.11	.3	68.98	63.36	61.13	54.38
.7	73.82	59.40	58.48	55.90	.5	53.70	49.68	50.12	49.61
.6	55.44	51.68	54.50	57.11	1.0	45.52	44.47	42.00	42.98
.5	50.10	52.03	50.01	49.38					

A	AE1	BE1	FE1	DE1	c	AE2	BE2	FE2	DE2
.9	69.94	89.54	76.12	77.58					
.8	63.17	72.77	63.55	62.56	.3	70.16	60.26	54.48	53.30
.7	61.75	57.09	56.56	52.55	.5	51.40	50.26	48.12	47.63
.6	57.02	55.58	52.23	48.96	1.0	45.95	43.68	41.53	42.02
.5	56.32	50.48	45.84	47.78					

Table 9. Mean Waiting Times of the Fifteenth Stage of a Series Queueing System

A	AA1	BA1	FA1	DA1	c	AA2	BA2	FA2	DA2
.9	871.53	1144.38	982.67	1027.50					
.8	378.23	374.34	360.09	464.42	.3	180.95	89.59	199.70	181.64
.7	270.62	254.27	264.28	261.76	.5	138.63	145.84	140.59	145.53
.6	189.15	183.93	171.81	197.78	1.0	95.20	103.87	102.01	197.07
.5	161.17	156.03	157.19	162.83					

A	AB1	BB1	FB1	DB1	c	AB2	BB2	FB2	DB2
.9	243.84	203.30	205.12	157.10					
.8	155.82	104.50	115.79	98.74	.3	90.80	85.99	87.61	76.47
.7	91.10	90.44	86.96	79.32	.5	73.80	70.98	69.15	65.52
.6	78.55	77.10	71.51	78.99	1.0	57.62	52.55	52.94	50.78
.5	72.63	68.86	70.18	66.58					

A	AC1	BC1	FC1	DC1	c	AC2	BC2	FC2	DC2
.9	108.23	121.38	110.04	101.76					
.8	82.37	85.40	68.53	64.91	.3	61.62	63.75	62.50	61.54
.7	60.85	73.22	65.61	66.57	.5	51.68	55.24	54.52	53.36
.6	54.92	56.50	56.02	52.50	1.0	44.54	47.08	45.84	41.75
.5	53.32	51.52	51.40	51.41					

A	AD1	BD1	FD1	DD1	c	AD2	BD2	FD2	DD2
.9	90.41	109.01	84.08	82.36					
.8	69.87	73.42	70.16	64.11	.3	63.53	64.51	59.73	55.10
.7	65.75	60.12	58.58	56.52	.5	50.00	51.84	50.75	52.51
.6	54.24	55.02	53.49	48.80	1.0	43.96	44.05	45.85	42.54
.5	50.11	50.92	50.48	48.10					

A	AE1	BE1	FE1	DE1	c	AE2	BE2	FE2	DE2
.9	68.35	97.26	83.78	79.44					
.8	58.86	67.90	59.11	65.13	.3	68.66	60.21	53.81	53.65
.7	57.18	56.20	57.18	55.61	.5	48.05	50.85	45.59	47.81
.6	57.72	53.50	53.67	49.39	1.0	43.78	43.01	42.52	41.19
.5	52.84	54.68	48.69	46.92					

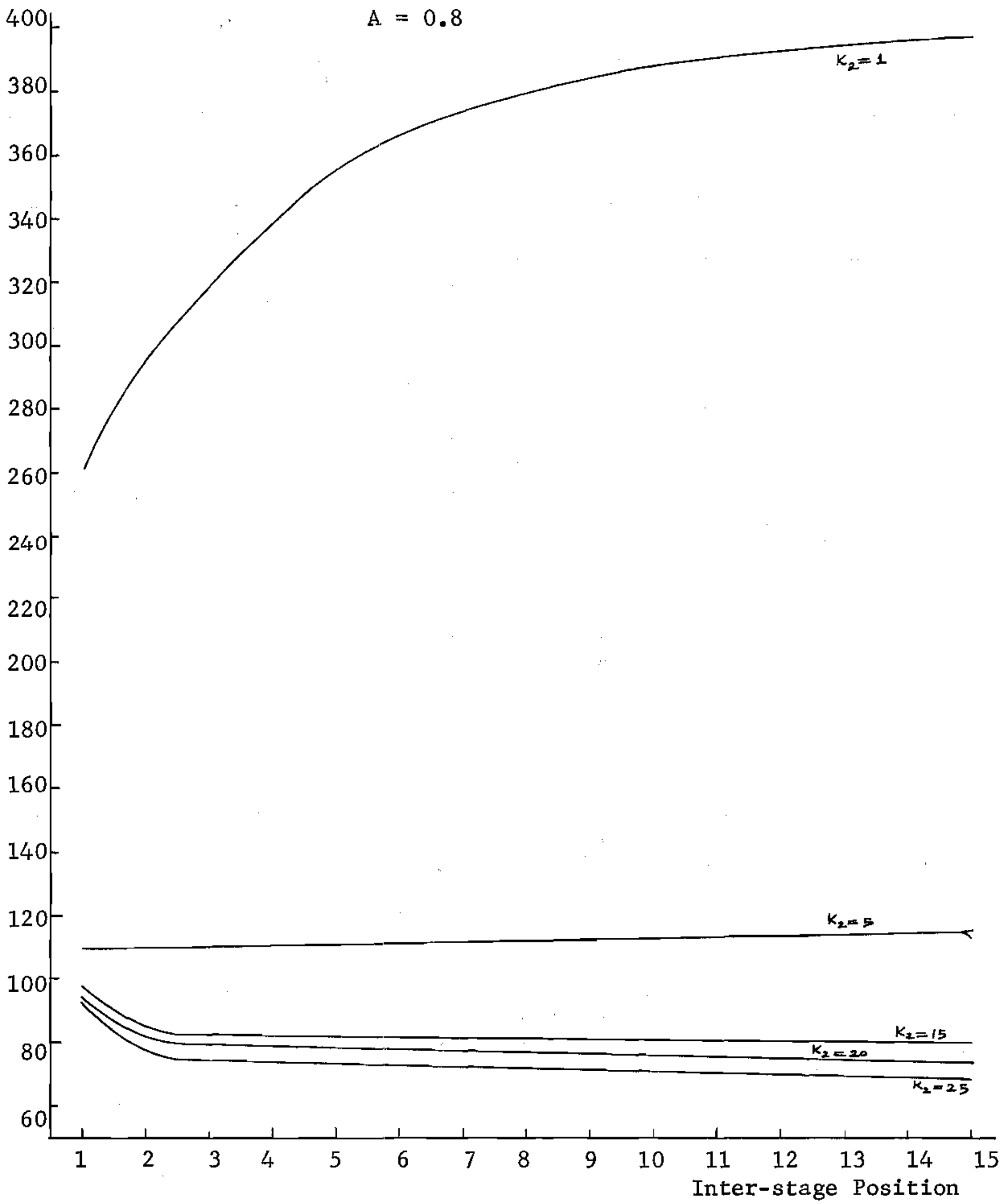

Figure 15. Mean Waiting Times Shown Against Inter-Stage Position - Model I - Erlangian Arrivals (K_1=5)

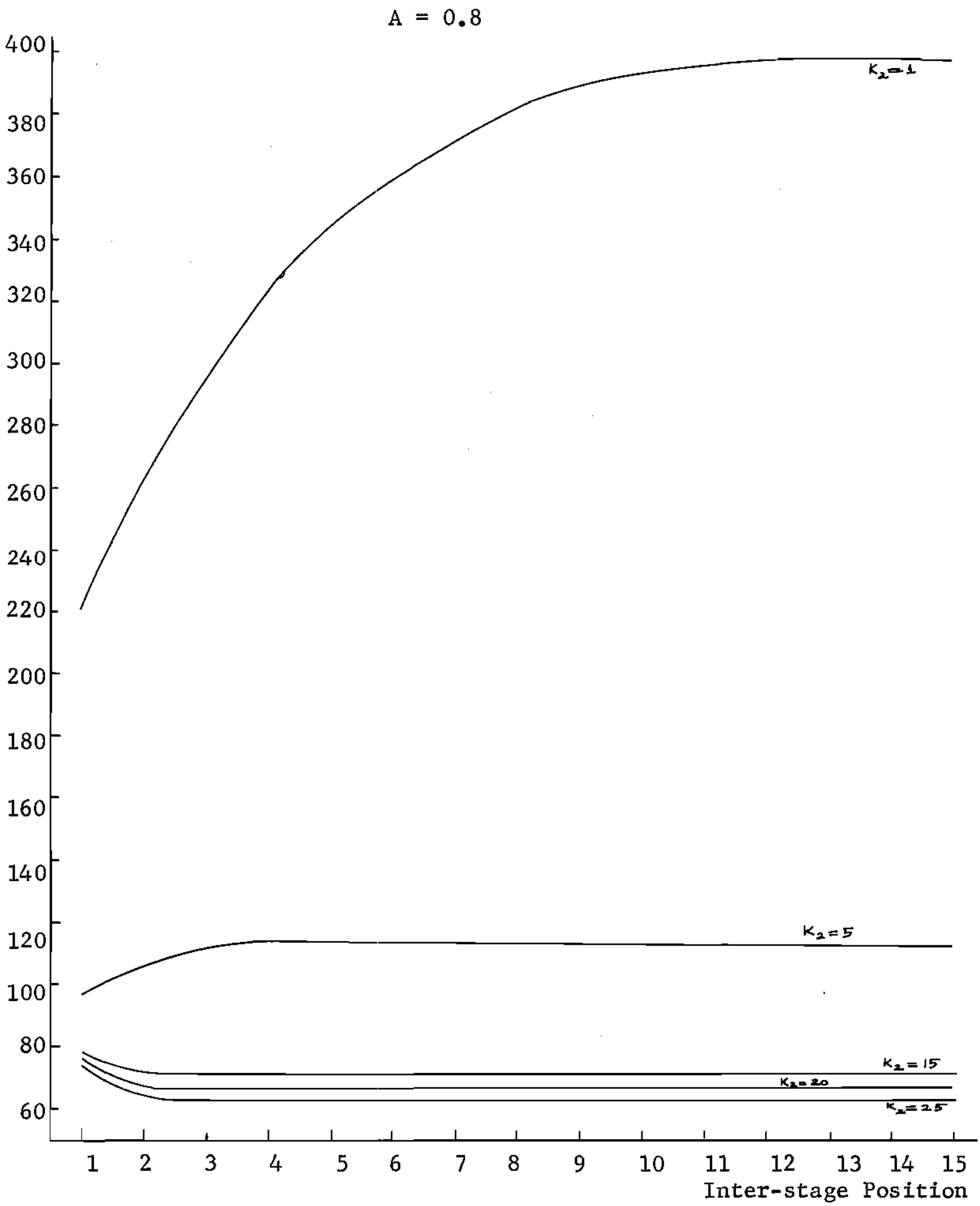

Figure 16. Mean Waiting Times Shown Against the Inter-Stage Position - Model I - Erlangian Arrivals (K_1 = 10)

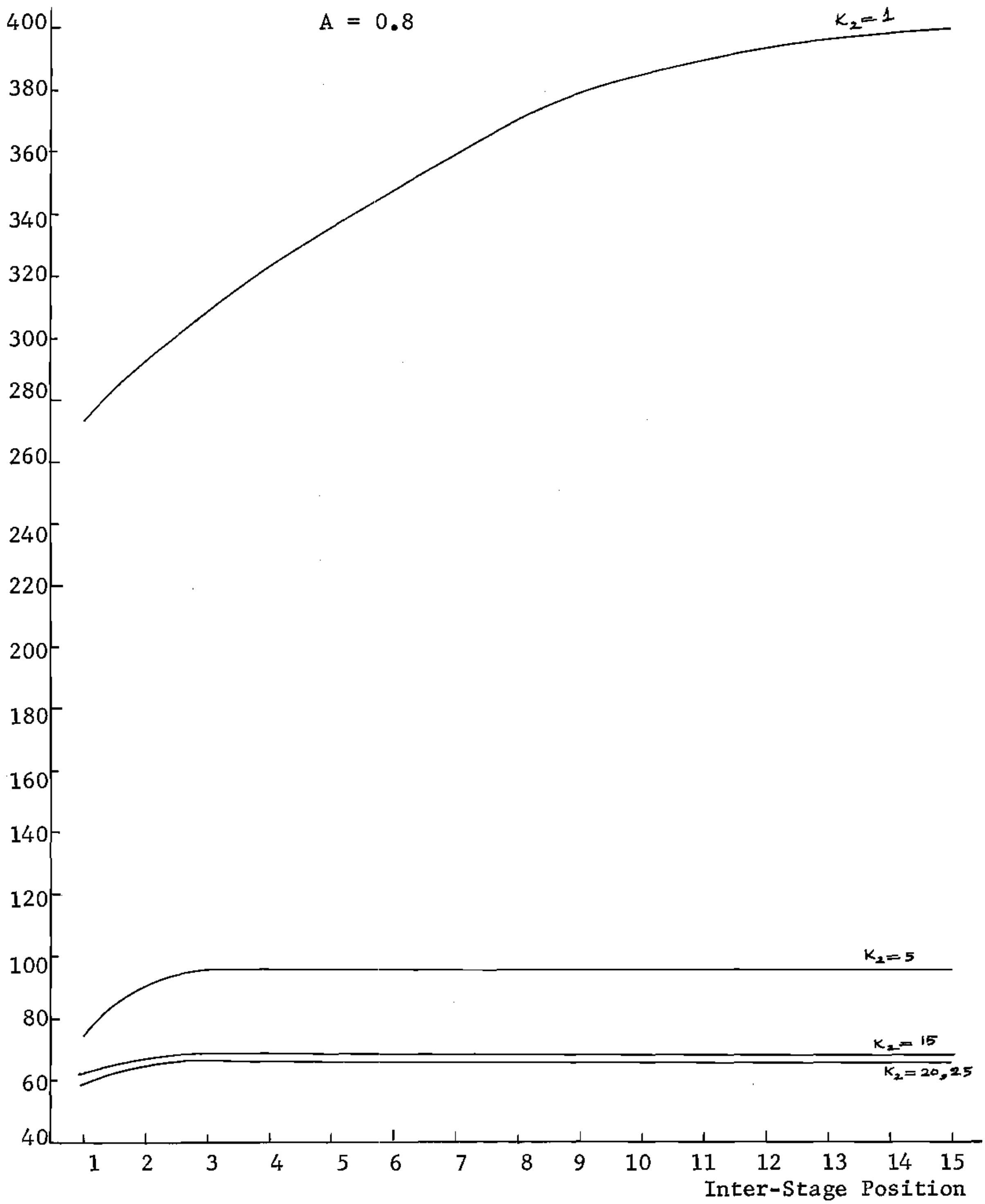

Figure 17. Mean Waiting Times Shown Against the Inter-Stage Position - Model I - Erlangian Arrivals (K_1=20)

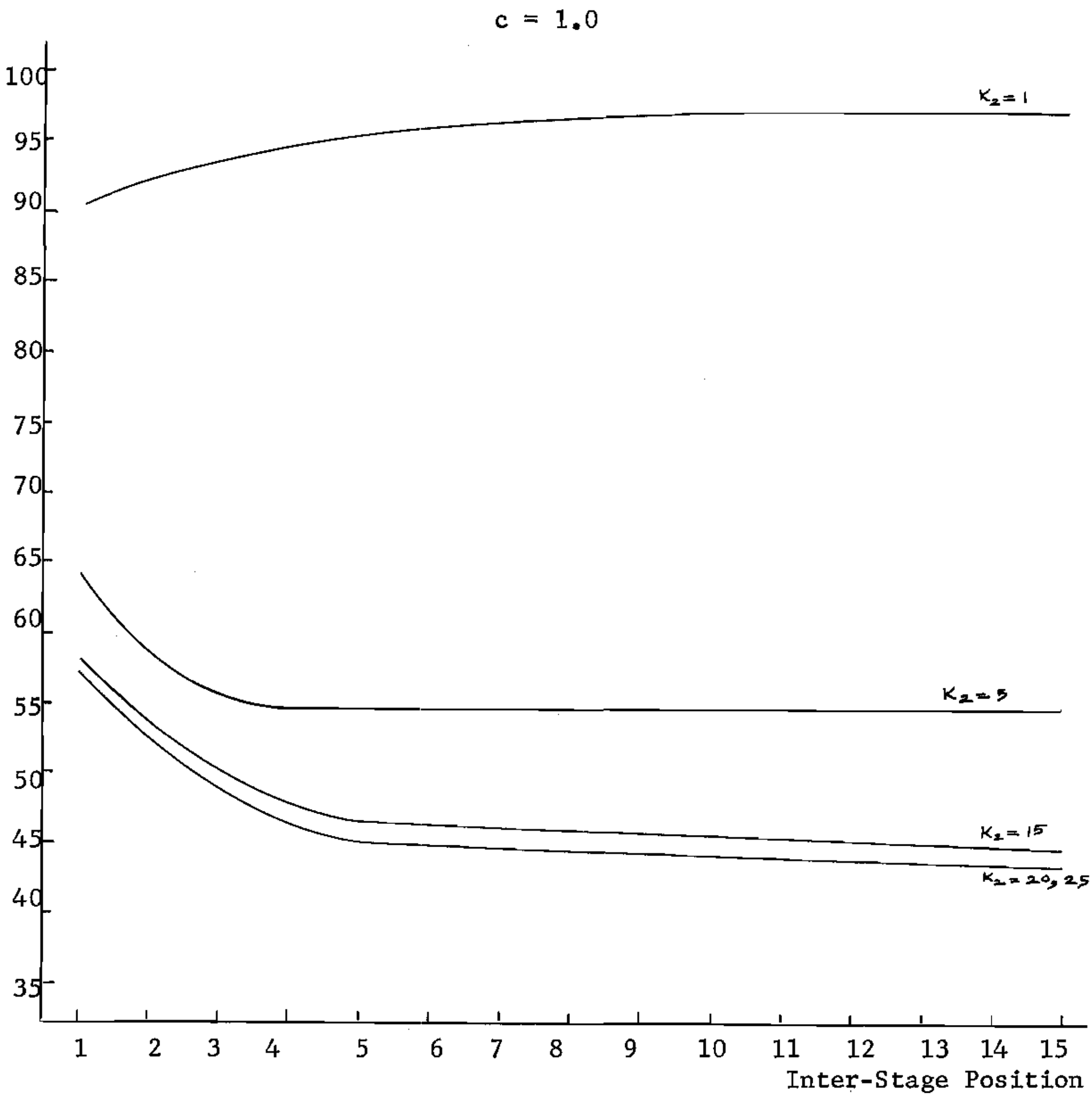

Figure 18. Mean Waiting Times Shown Against the Inter-Stage Position - Model II - Poisson Arrivals (K_1=1)

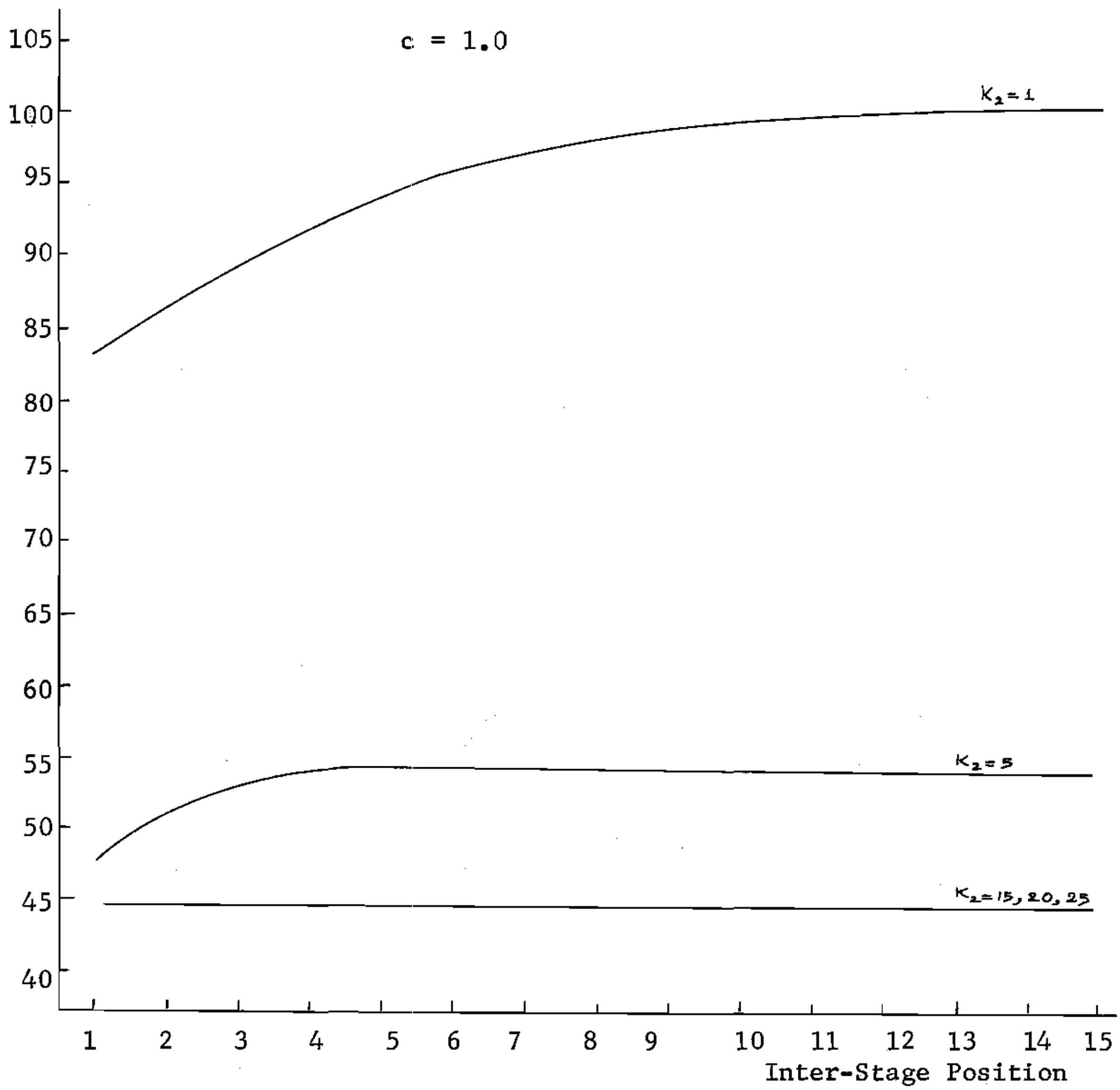

Figure 19. Mean Waiting Time Against the Inter-Stage Position-
Model II - Erlangian Arrivals ($K_1 = 5$)

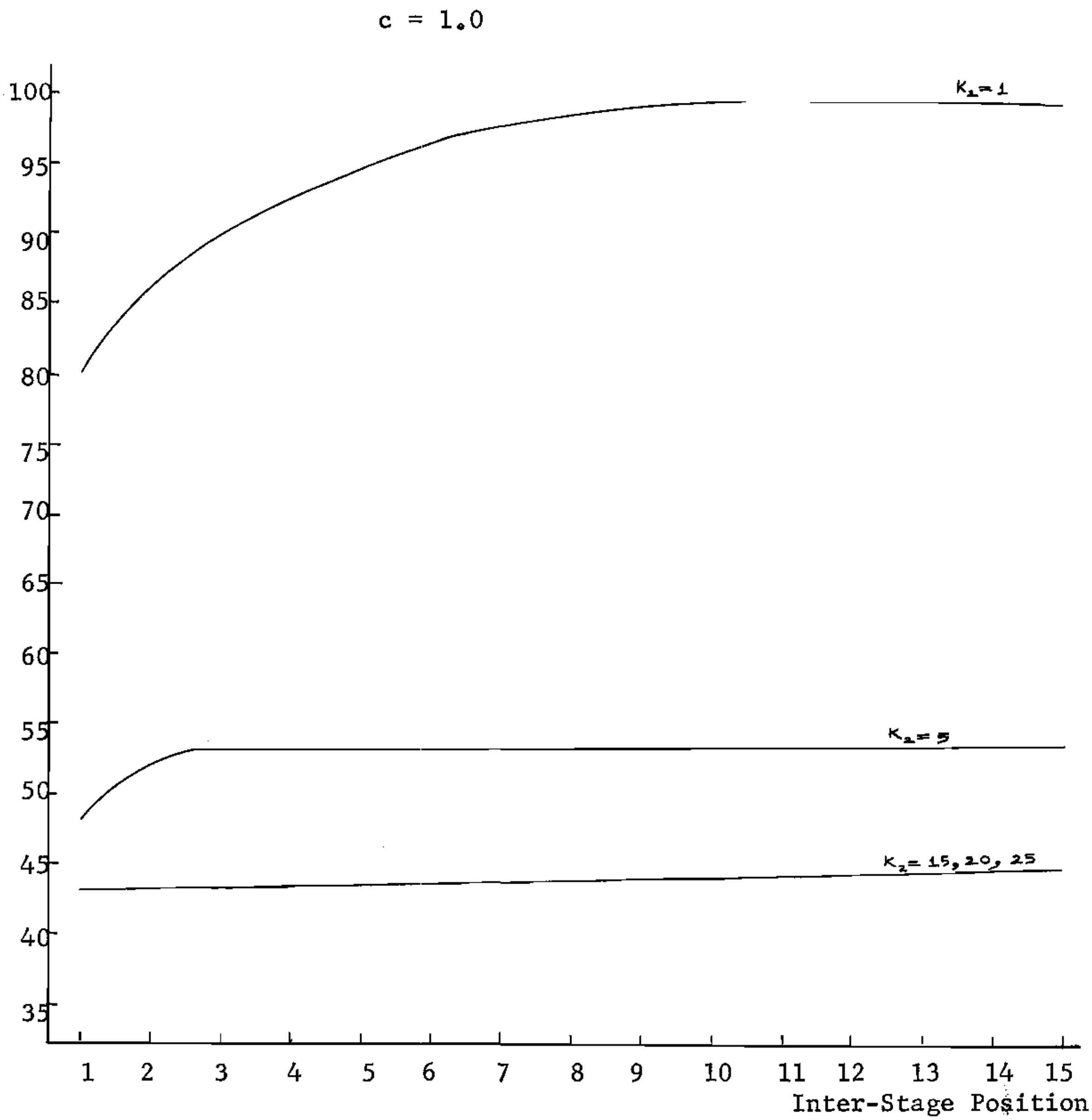

Figure 20. Mean Waiting Times Shown Against the Inter-Stage Position - Model II - Erlangian Arrivals (K_1=10)

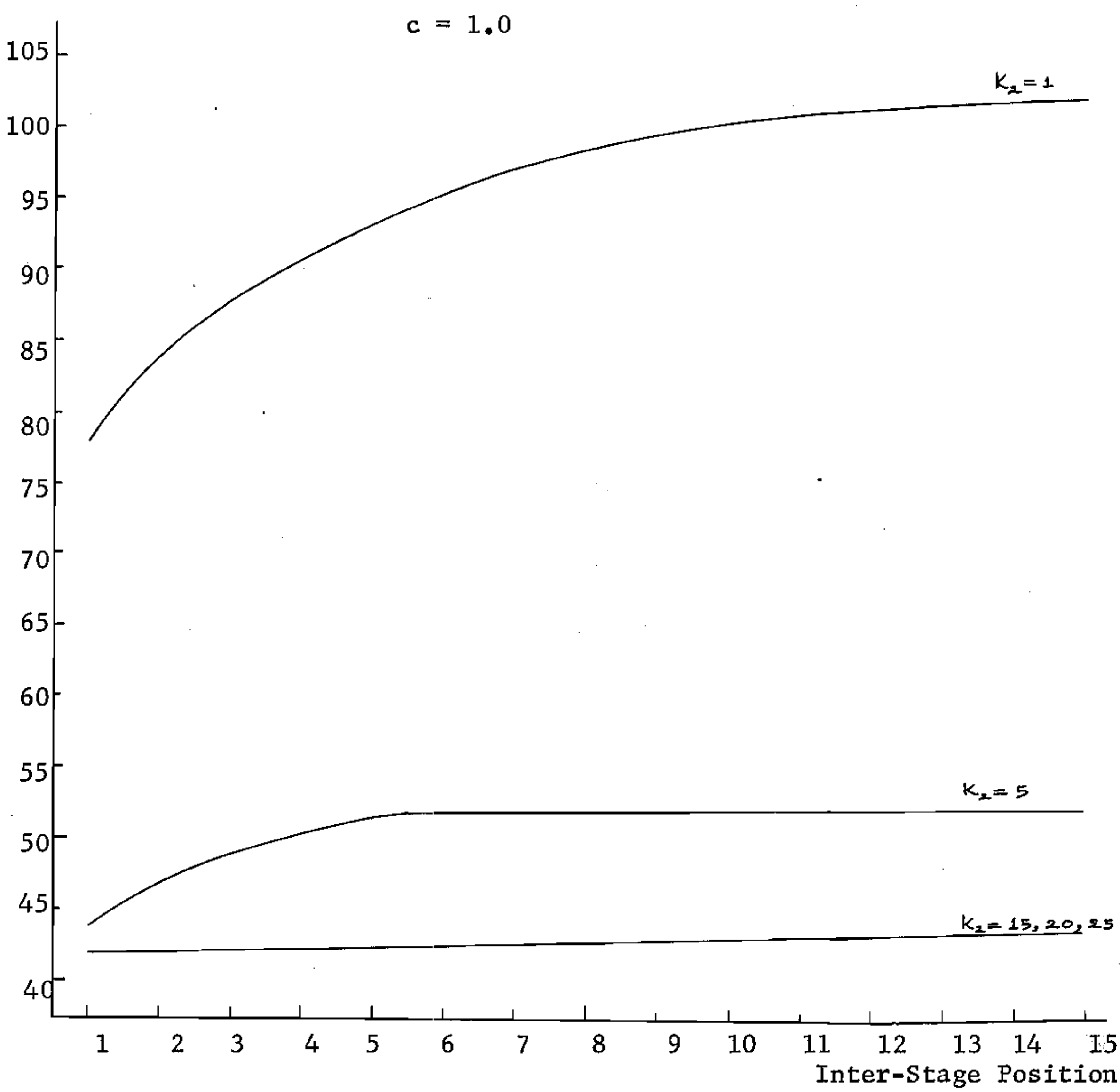

Figure 21. Mean Waiting Time Against the Inter-Stage Position-Model II - Erlangian Arrivals (K_1=20)

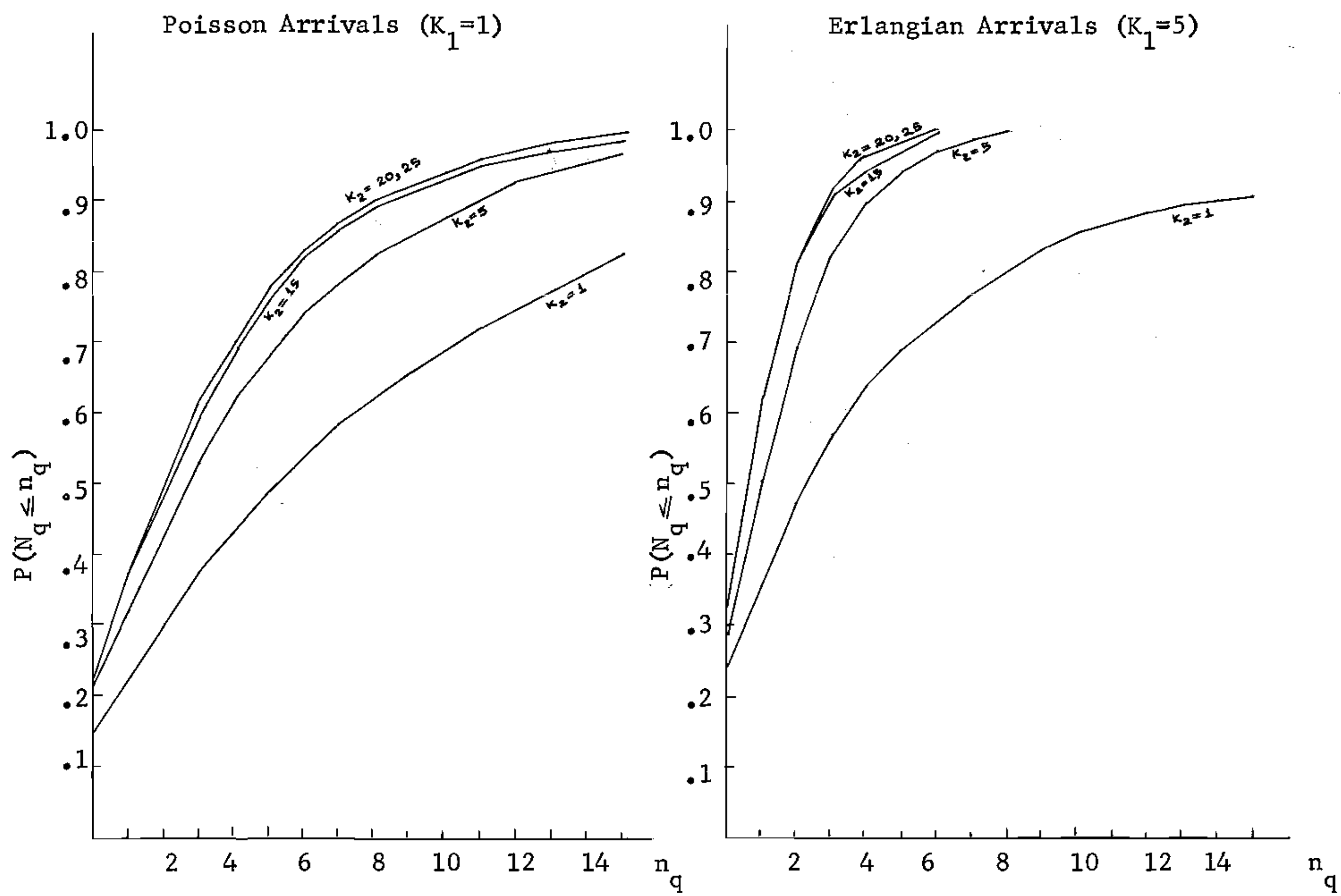

Figure 22. The State Probability Distributions at the First Stage - Model I (A=0.9) (continued on next page)

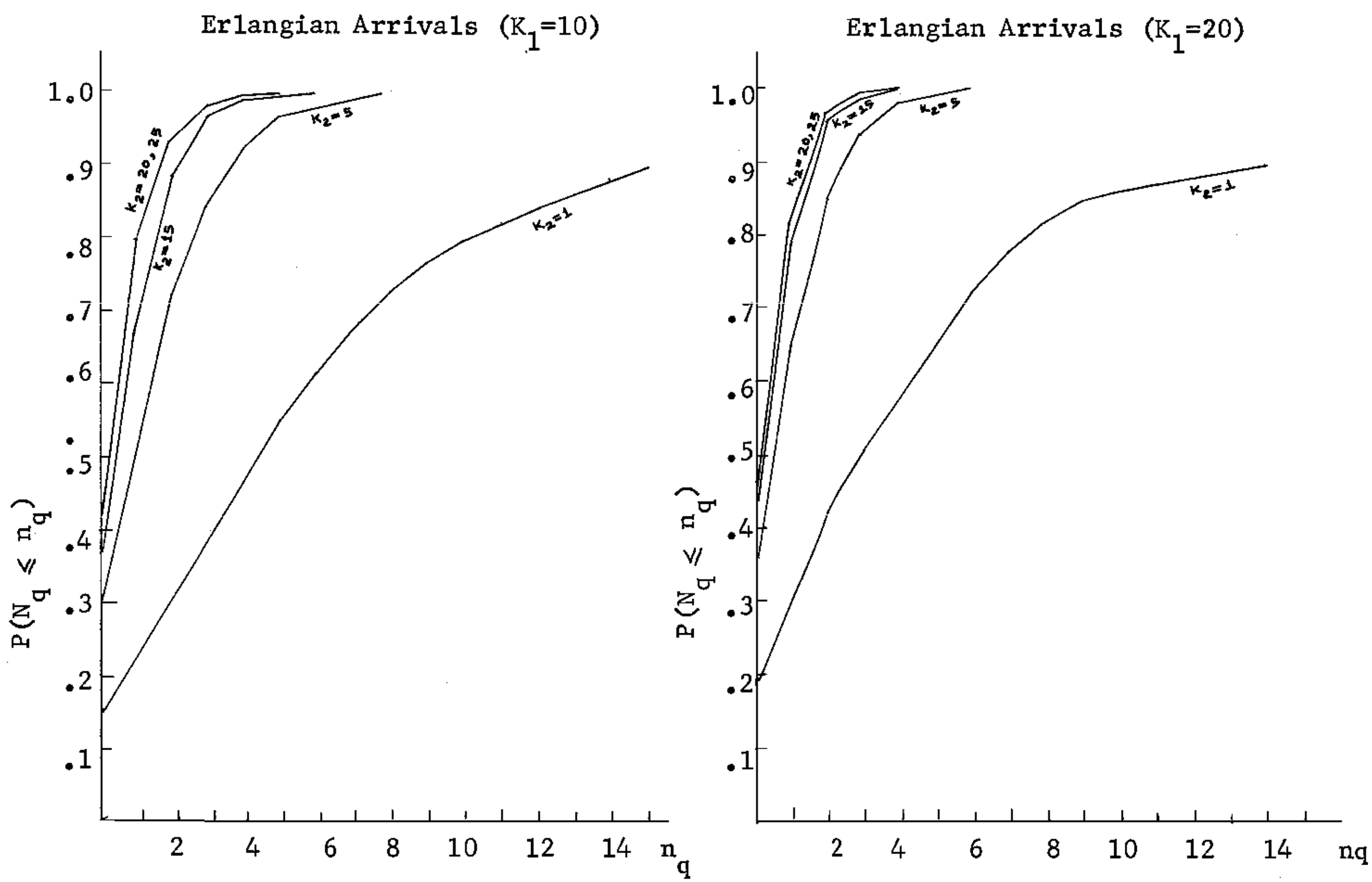

Figure 22 Continued

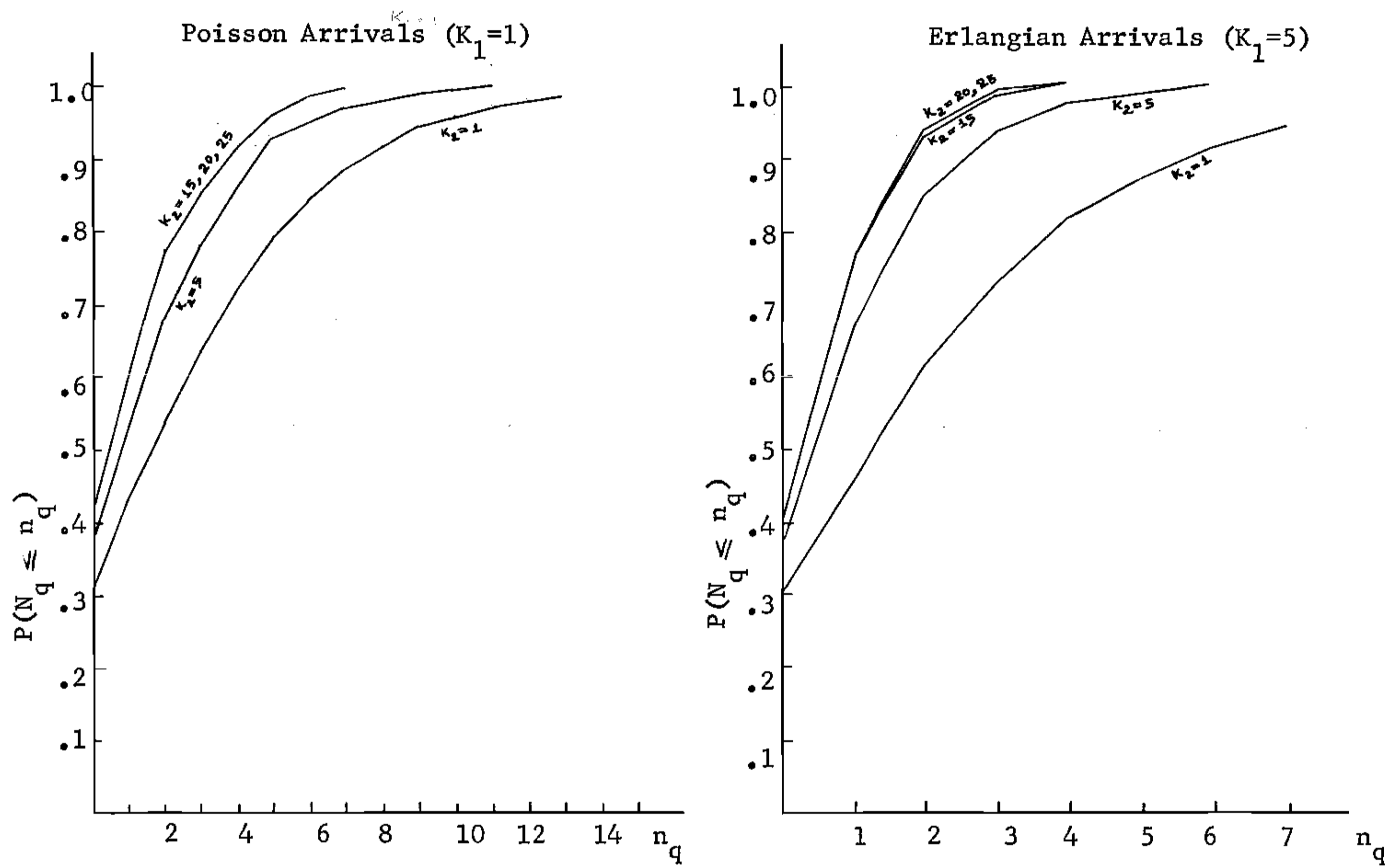

Figure 23. The State Probability Distributions at the First Stage - Model I (A=0.8) (continued on next page)

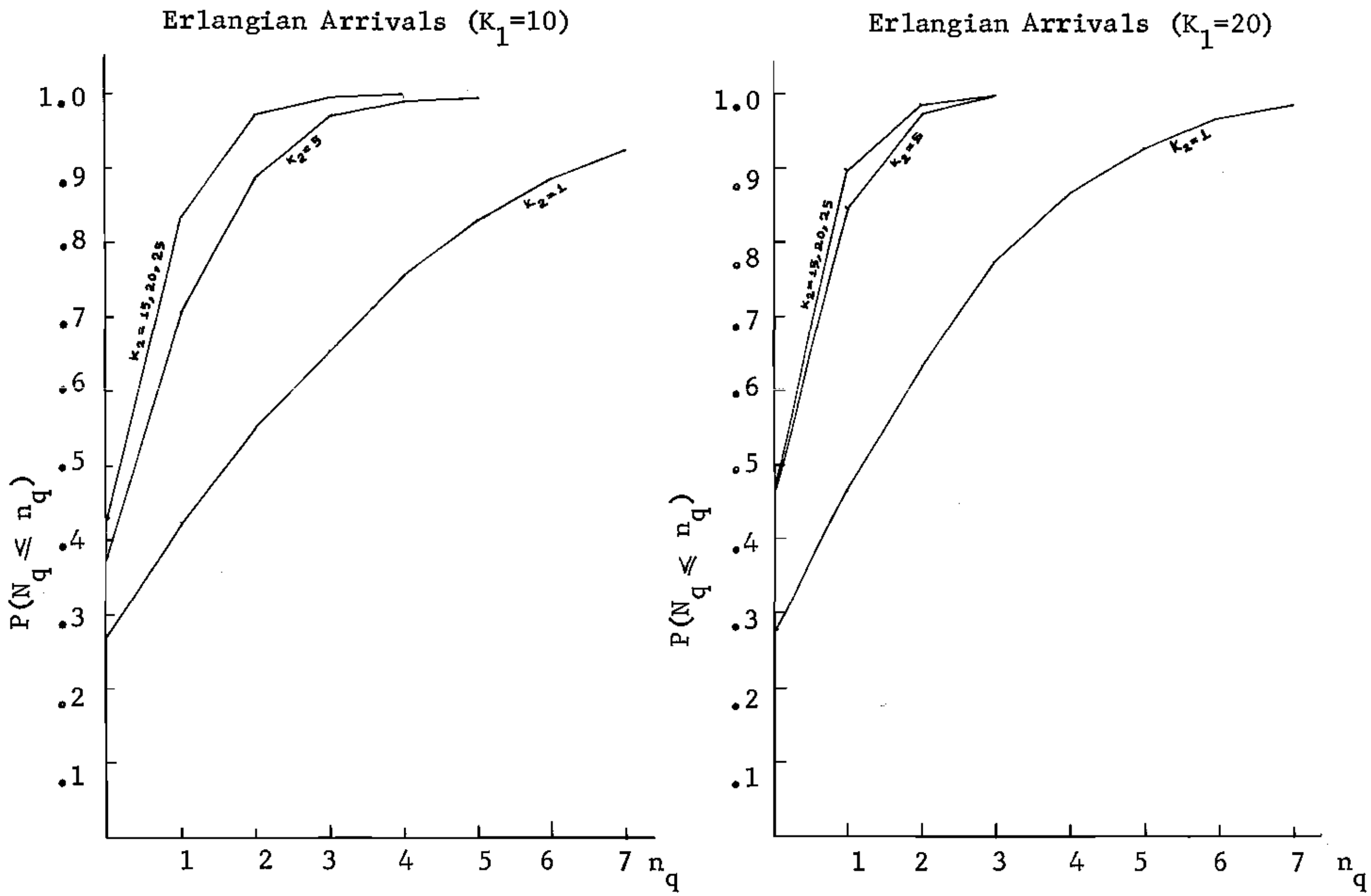

Figure 23 Continued

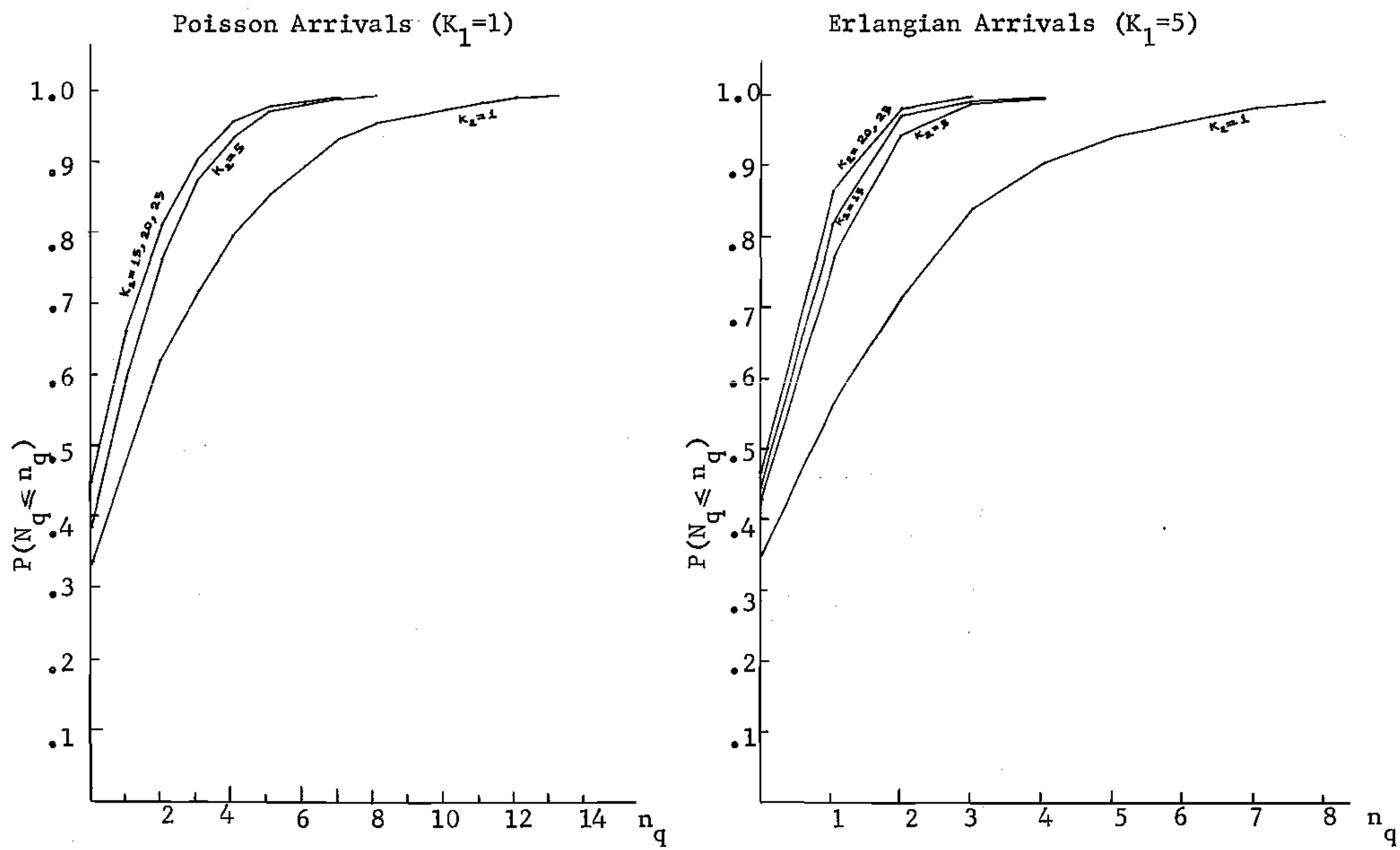

Figure 24. The State Probability Distributions at the First Stage - Model I
(A=0.7) (continued on next page)

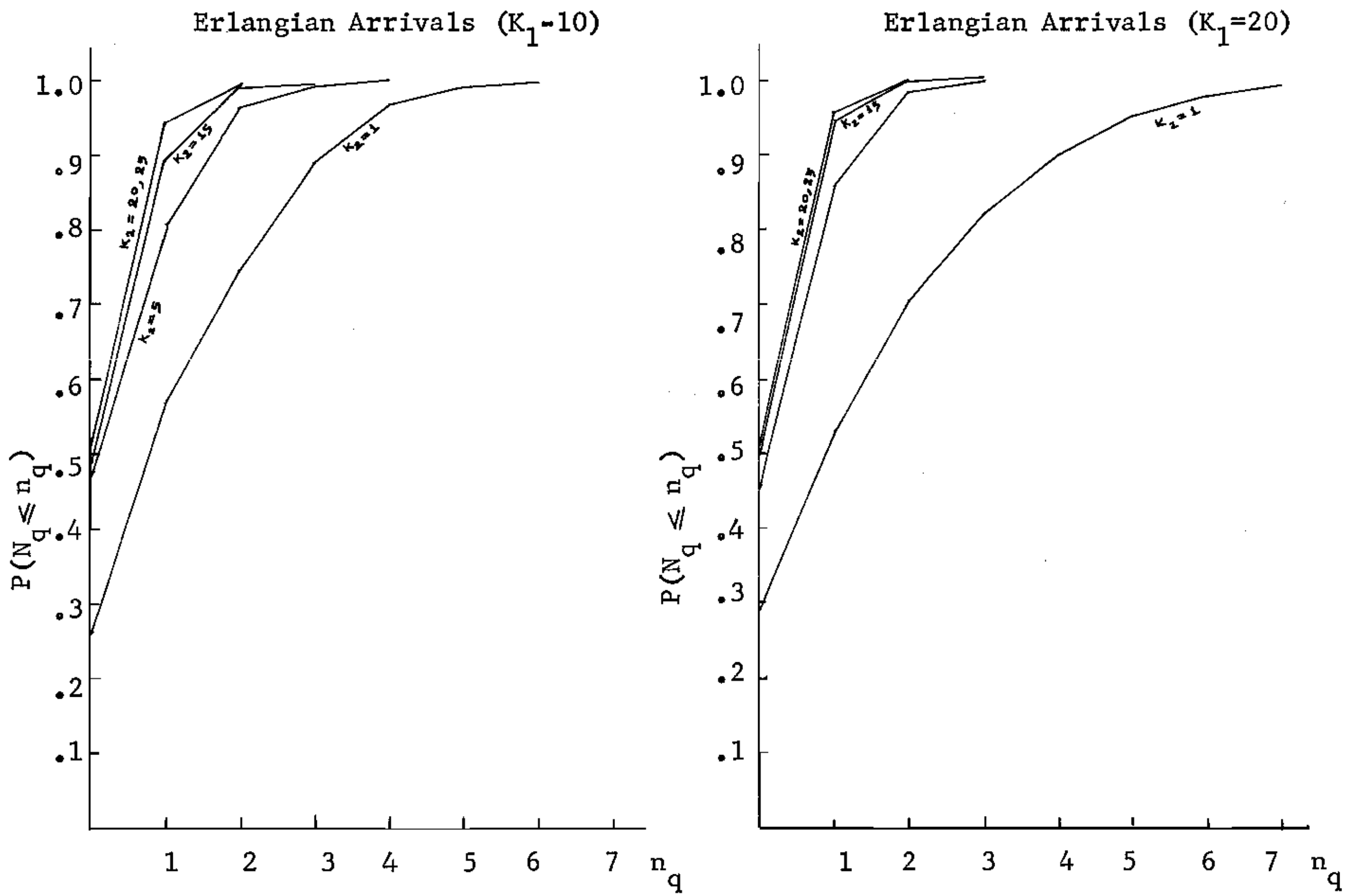

Figure 24 Continued

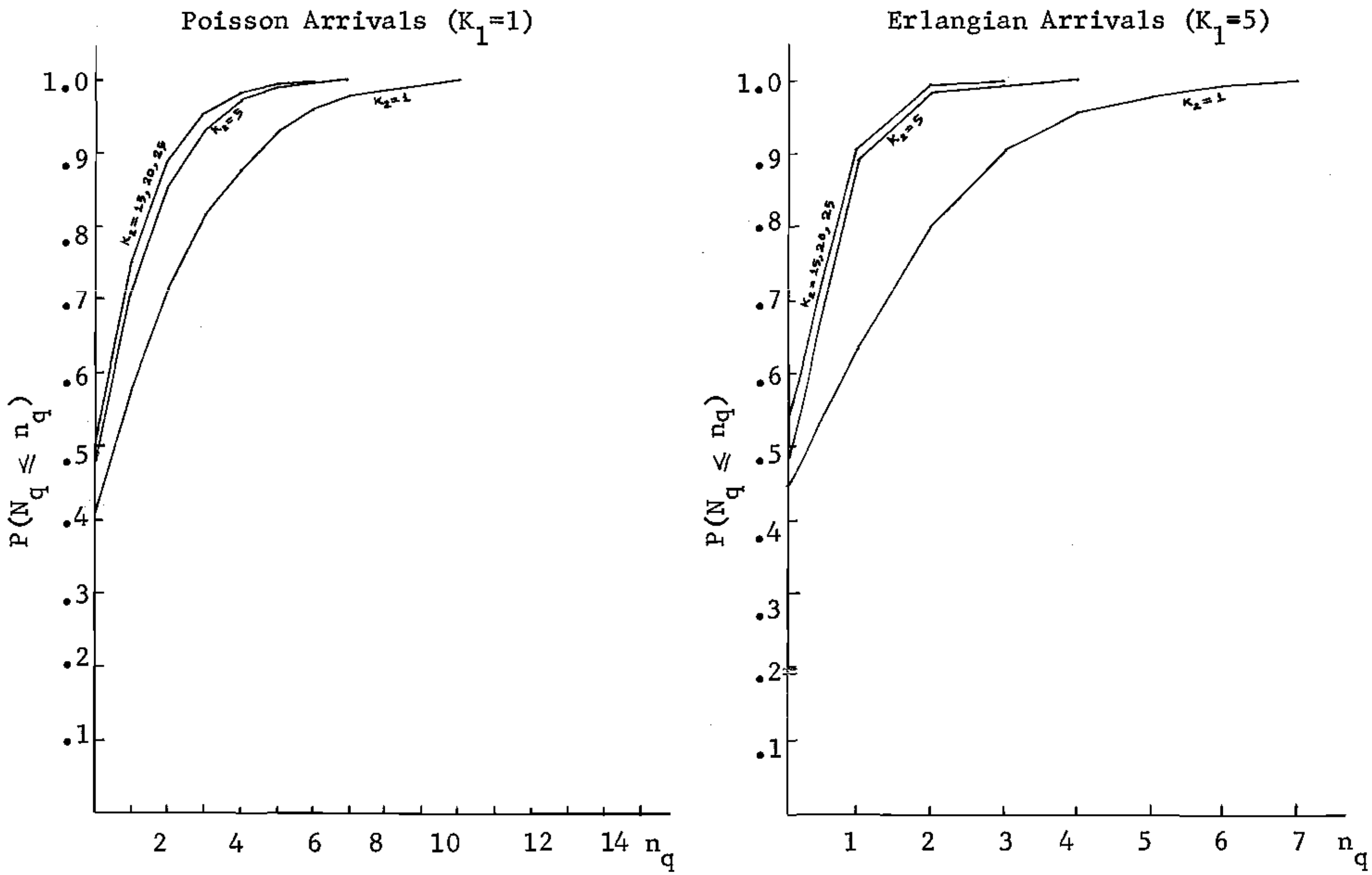

Figure 25. The State Probability Distributions at the First Stage - Model I - (A=0.6) (continued on next page)

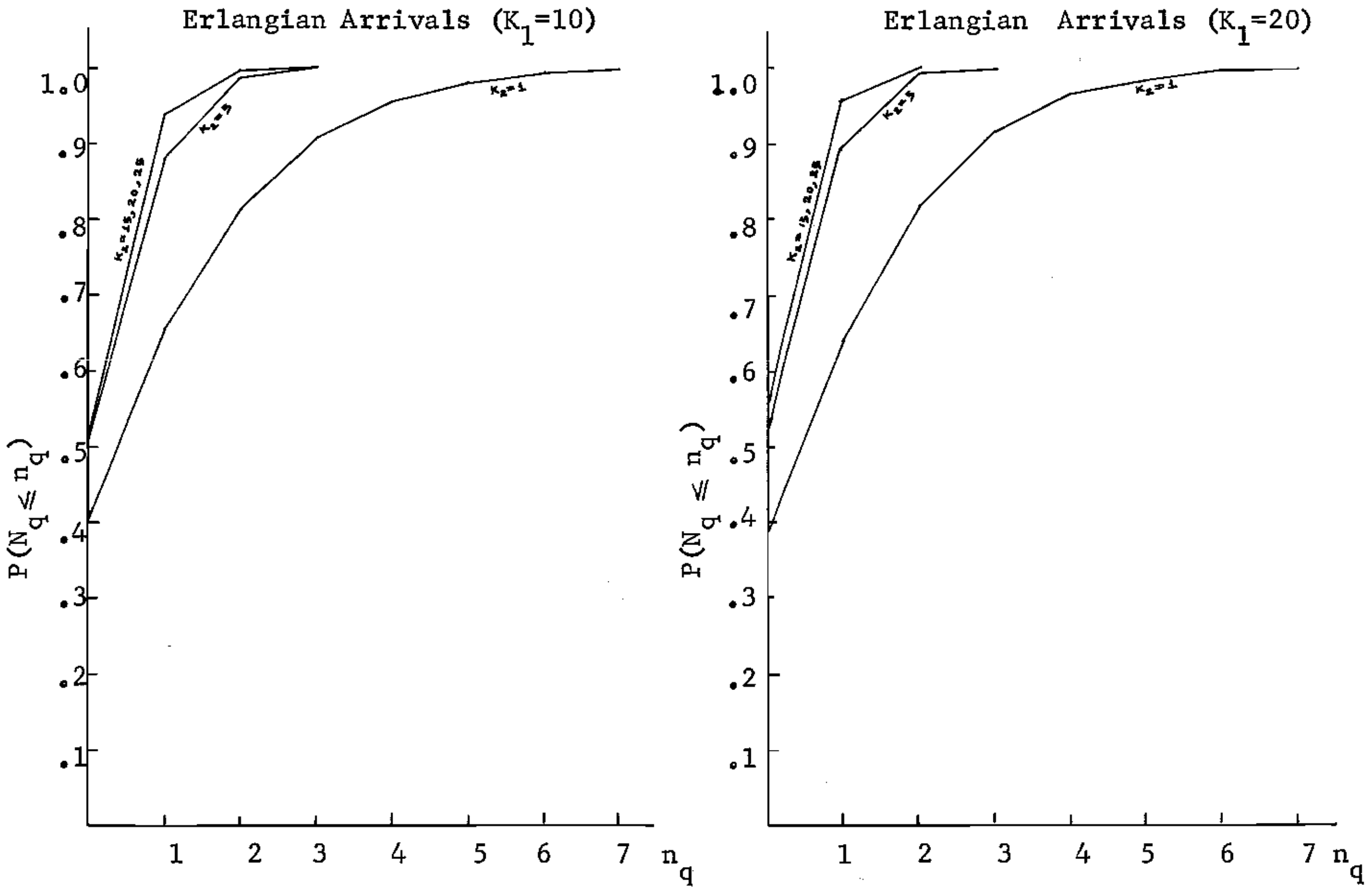

Figure 25 Continued

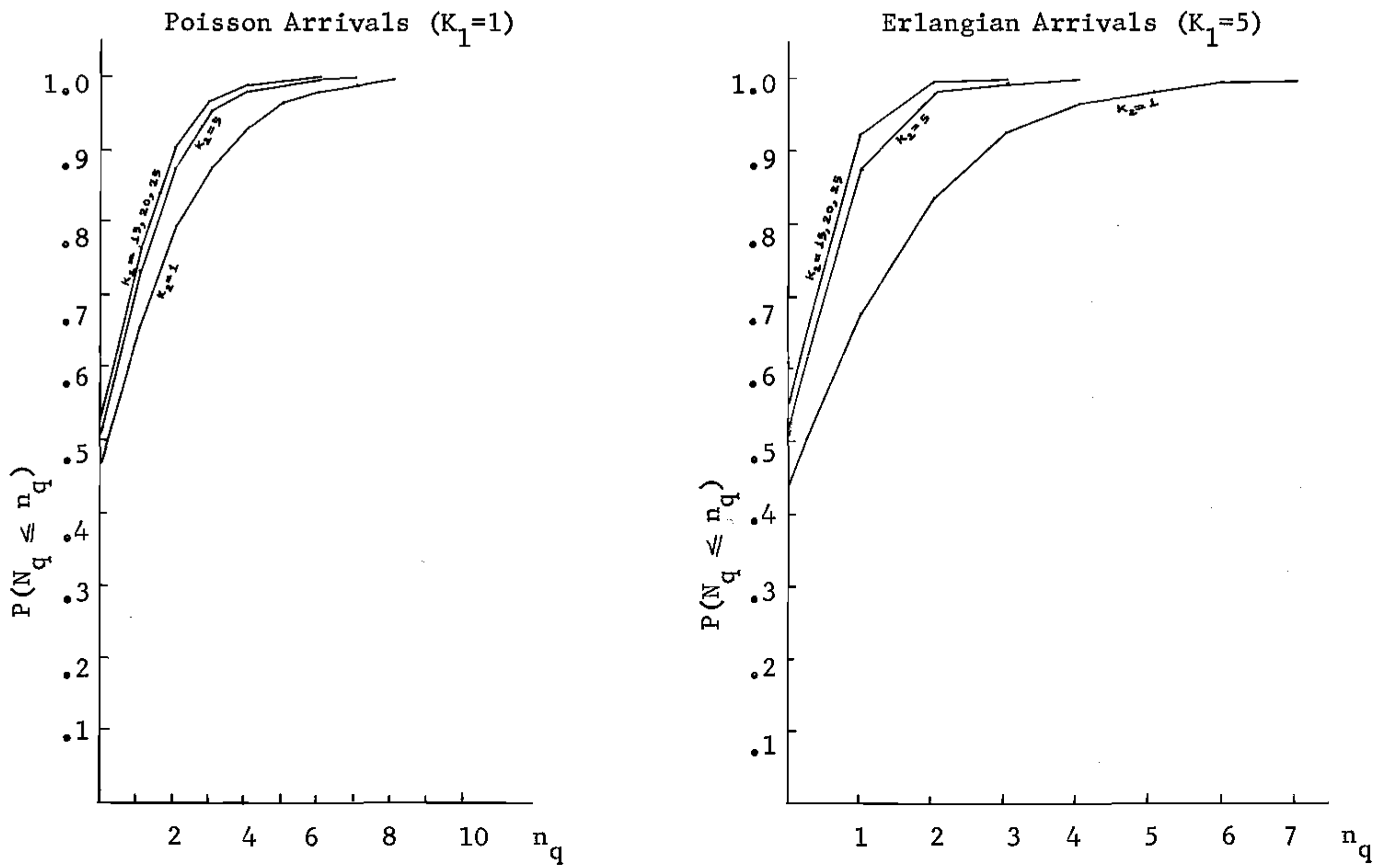

Figure 26. The State Probability Distributions at the First Stage - Model II - (A=0.5) (continued on next page)

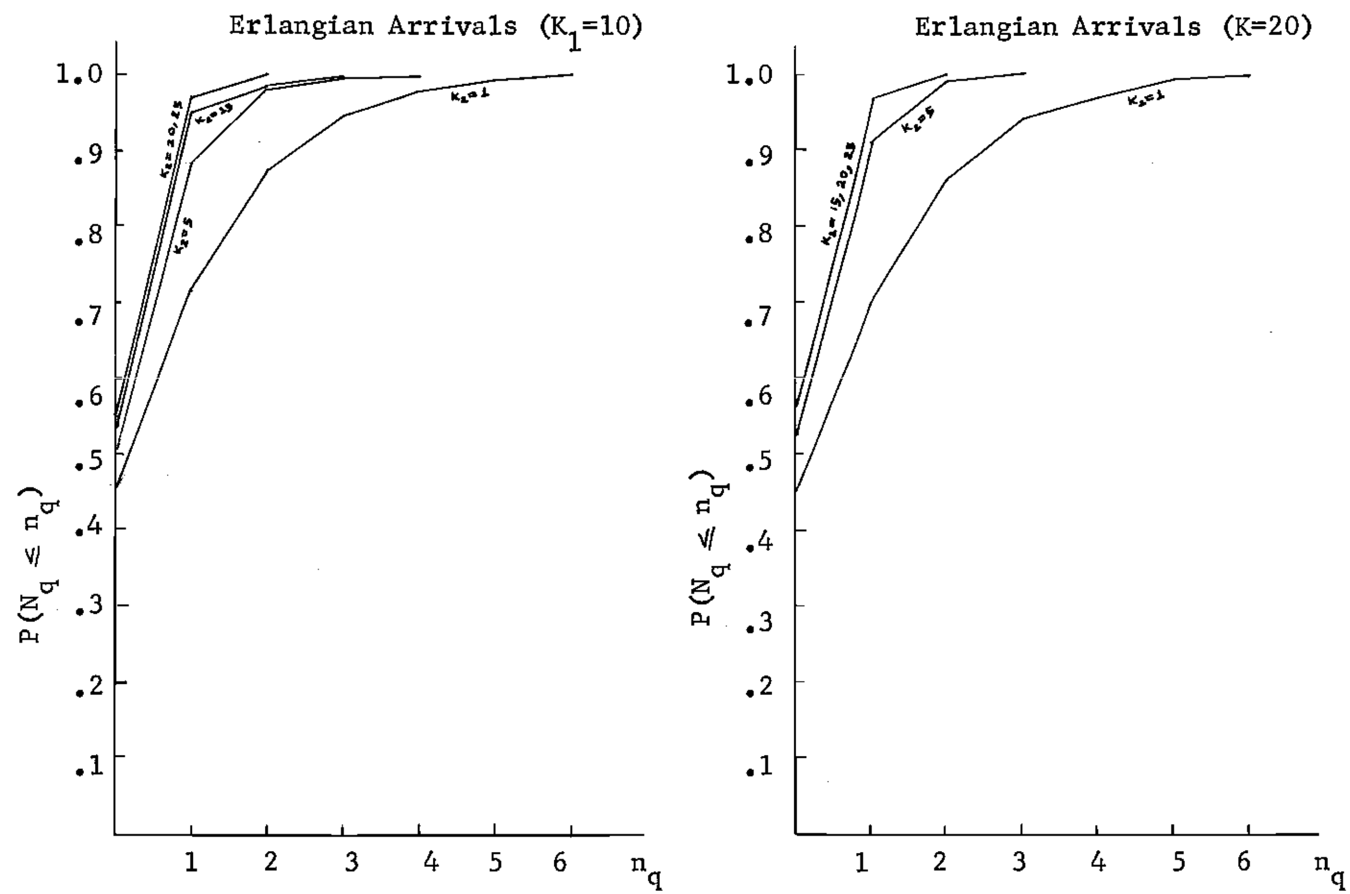

Figure 26 Continued

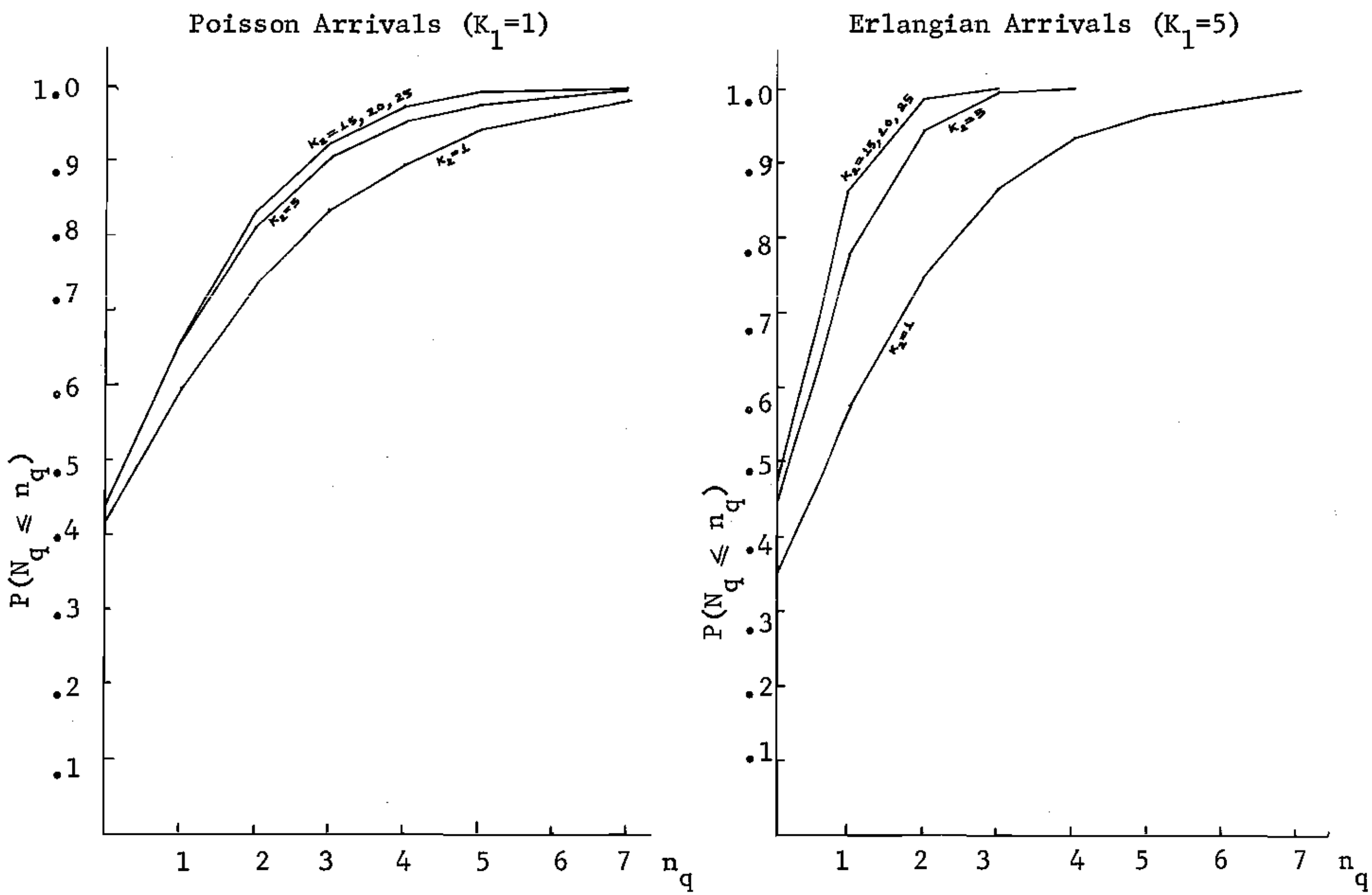

Figure 27. The State Probability Distributions of the First Stage - Model II (c=0.3) (continued on next page)

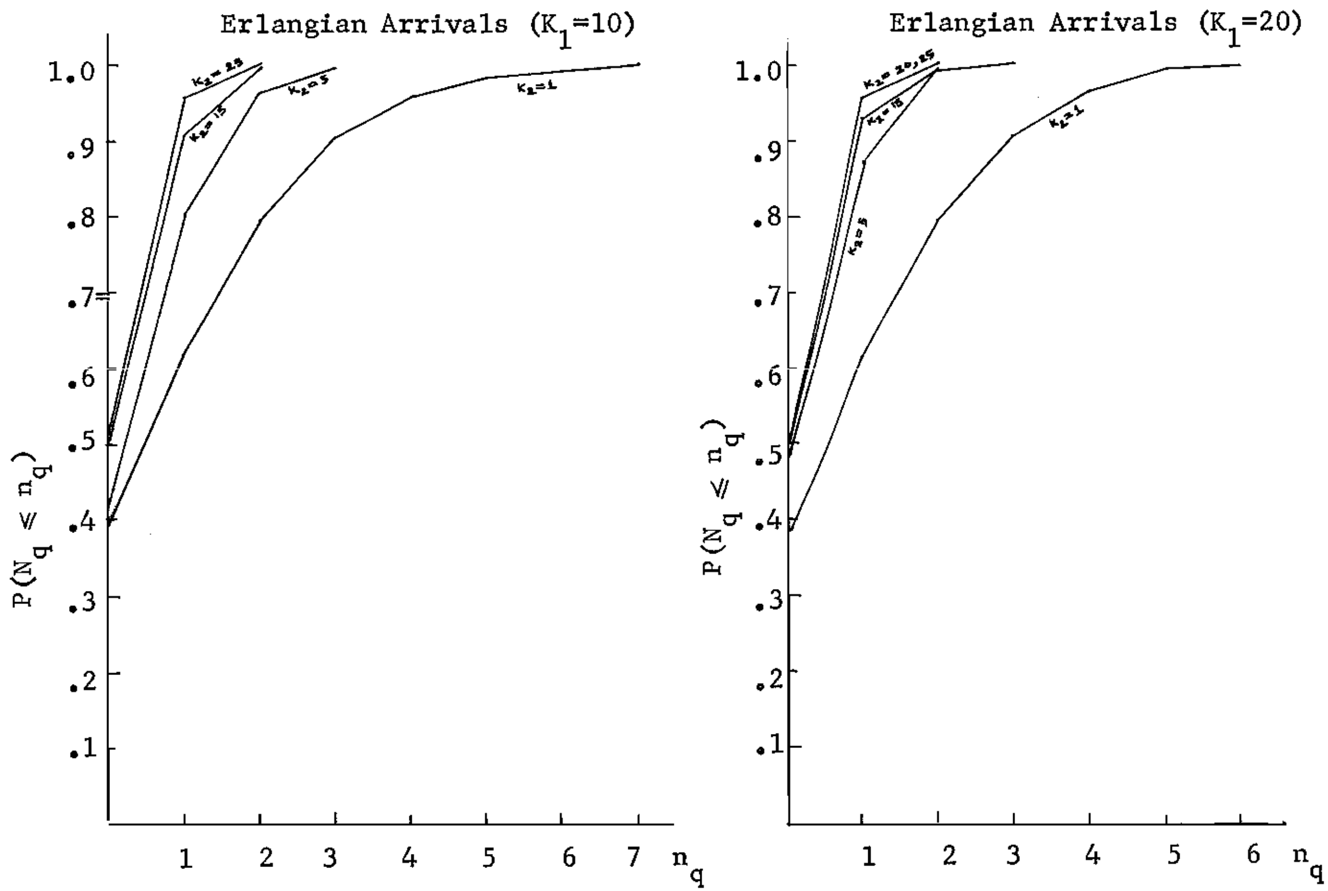

Figure 27 Continued

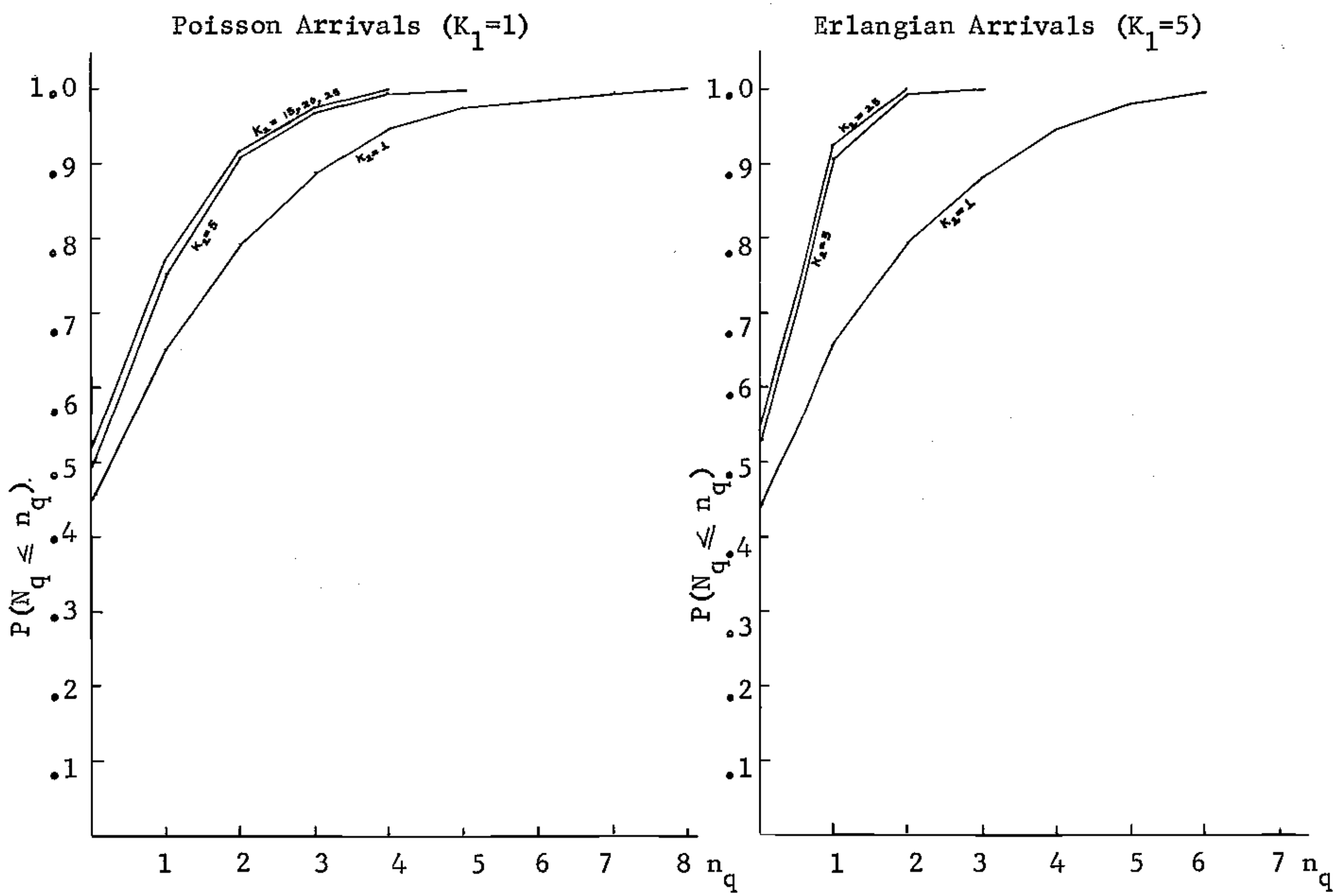

Figure 28. The State Probability Distributions at the First Stage - Model II (c=0.5)

(continued on next page)

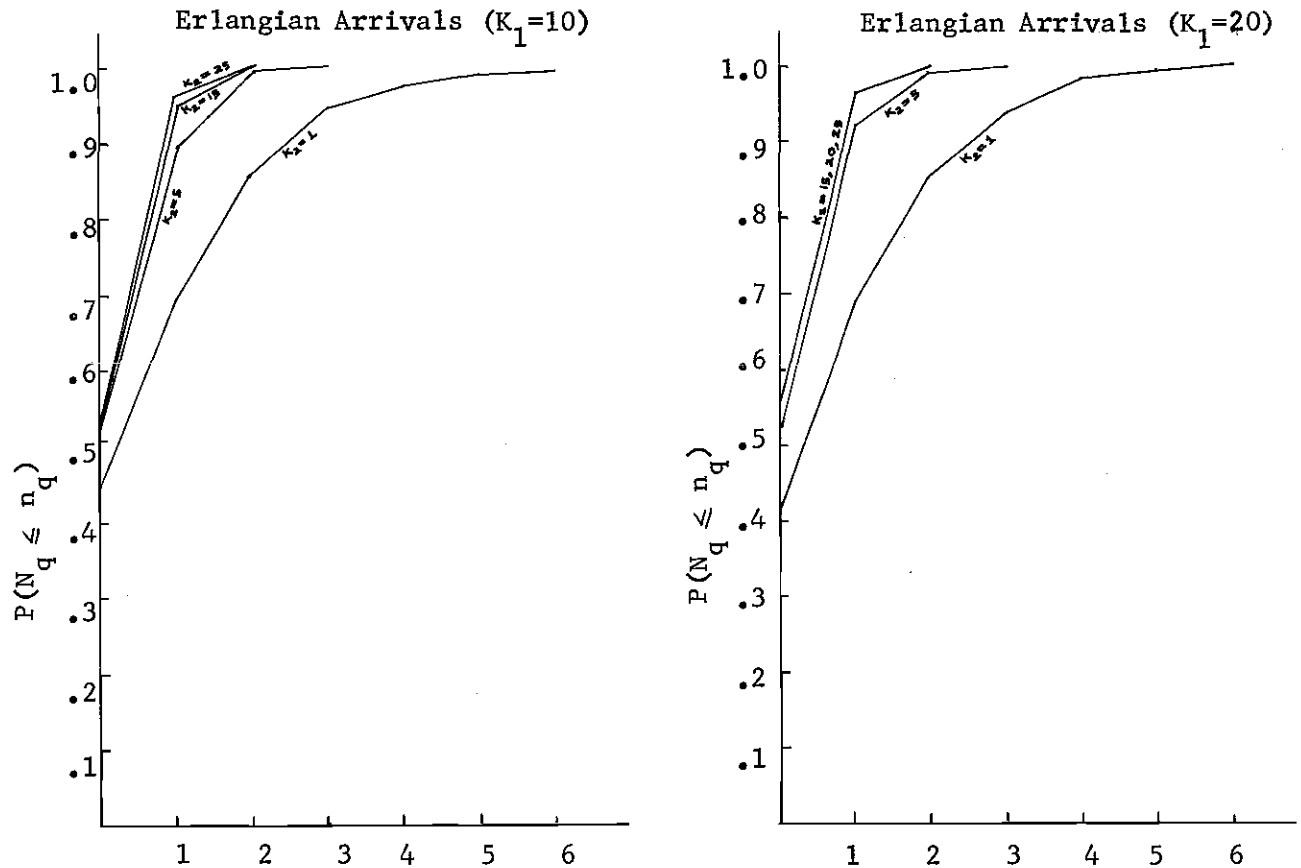

Figure 28 Continued

Table 10. Average Utilization of the Facility

Stage 1

Parameter K_1	Parameter K_2	A=0.9 B=0.1	A=0.8 B=0.2	A=0.7 B=0.3	A=0.6 B=0.4	A=0.5 B=0.5	C=0.3	C=0.5	C=1.0
	1	.93	.85	.84	.81	.76	.79	.77	.71
	5	.92	.88	.85	.81	.79	.81	.80	.77
1	15	.91	.81	.82	.81	.79	.82	.79	.75
	20	.92	.87	.85	.81	.79	.84	.79	.78
	25	.90	.86	.84	.83	.82	.85	.80	.78
	1	.91	.89	.87	.84	.83	.87	.83	.80
	5	.95	.92	.92	.90	.89	.91	.90	.87
5	15	.96	.95	.94	.94	.92	.93	.92	.91
	20	.97	.95	.94	.93	.93	.94	.92	.92
	25	.97	.95	.94	.93	.93	.94	.93	.93
	1	.94	.90	.88	.85	.84	.86	.83	.81
	5	.96	.94	.92	.92	.92	.93	.92	.91
10	15	.97	.95	.95	.95	.94	.95	.95	.93
	20	.97	.96	.95	.95	.95	.95	.95	.95
	25	.97	.96	.96.	.96	.95	.95	.95	.95
	1	.93	.91	.90	.87	.84	.86	.86	.82
	5	.96	.92	.94	.93	.92	.93	.92	.91
20	15	.97	.96	.96	.95	.95	.96	.95	.95
	20	.97	.97	.97	.96	.96	.96	.96	.96
	25	.98	.97	.96	.96	.96	.97	.96	.96

Table 11. Average Utilization of the Facility

Stage 5

Parameter K_1	Parameter K_2	A=0.9 B=0.1	A=0.8 B=0.2	A=0.7 B=0.3	A=0.6 B=0.4	A=0.5 B=0.5	C=0.3	C=0.5	C=1.0
	1	.92	.86	.82	.80	.75	.78	.75	.67
	5	.93	.91	.89	.85	.85	.86	.84	.80
1	15	.95	.92	.88	.88	.86	.88	.87	.83
	20	.95	.92	.90	.89	.87	.90	.86	.87
	25	.93	.90	.90	.91	.90	.92	.88	.87
	1	.92	.87	.83	.79	.76	.81	.75	.68
	5	.95	.92	.91	.89	.89	.91	.88	.84
5	15	.97	.96	.95	.94	.93	.95	.93	.92
	20	.97	.96	.96	.95	.95	.96	.95	.94
	25	.98	.97	.96	.96	.96	.97	.96	.95
	1	.92	.87	.83	.80	.78	.82	.76	.69
	5	.95	.93	.90	.89	.88	.91	.89	.85
10	15	.97	.95	.95	.94	.93	.95	.94	.93
	20	.97	.97	.96	.95	.95	.96	.95	.95
	25	.97	.97	.97	.96	..95	.96	.95	.95
	1	.92	.87	.82	.82	.78	.81	.76	.69
	5	.94	.92	.91	.90	.88	.91	.89	.86
20	15	.97	.96		.94	.95	.95	.95	.93
	20	.97	.96	.96	.94	.96	.96	.95	.95
	25	.98	.97	.97	.96	.96	.97	.95	

Table 12. Average Utilization of the Facility

Stage 10

Parameter K_1	Parameter K_2	A=0.9 B=0.1	A=0.8 B=0.2	A=0.7 B=0.3	A=0.6 B=0.4	A=0.5 B=0.5	C=0.3	C=0.5	C=1.0
	1	.91	.87	.82	.79	.75	.78	.75	.65
	5	.94	.91	.90	.86	.85	.88	.85	.81
1	15	.94	.93	.90	.89	.88	.90	.89	.86
	20	.95	.94	.92	.91	.89	.92	.88	.90
	25	.94	.91	.92	.92	.92	.94	.90	.90
	1	.92	.88	.82	.79	.76	.80	.75	.66
	5	.95	.92	.90	.89	.89	.90	.88	.84
5	15	.97	.96	.96	.94	.95	.95	.94	.92
	20	.98	.97	.96	.95	.96	.96	.95	.94
	25	.98	.97	.96	.96		.97	.96	.95
	1	.91	.86	.83	.79	.76	.80	.75	.65
	5	.95	.92	.90	.89	.88	.91	.88	.85
10	15	.97	.96	.95	.94	.93	.96	.94	.93
	20	.97	.97	.96	.96	.95	.96	.95	.95
	25	.97	.97	.97	.96	.96	.96	.96	.95
	1	.91	.87	.84	.78	.78	.81	.75	.65
	5	.94	.91	.90	.89	.89	.91	.89	.85
20	15	.97	.95		.94	.94	.95	.94	.93
	20	.97	.96	.96	.95	.95	.96	.95	.95
	25	.98	.97	.97	.96	.96	.97	.96	

Table 13. Average Utilization of the Facility

Stage 15

Parameter K_1	Parameter K_2	A=0.9 B=0.1	A=0.8 B=0.2	A=0.7 B=0.3	A=0.6 B=0.4	A=0.5 B=0.5	C=0.3	C=0.5	C=1.0
	1	.91	.87	.81	.78	.75	.78	.75	.63
1	5	.94	.92	.90	.86	.85	.88	.86	.81
	15	.95	.94	.90	.90	.89	.90	.89	.87
	20	.95	.95	.92	.91	.89	.93	.89	.91
	25	.94	.92	.92	.92	.93	.95	.91	.91
	1	.92	.86	.82	.79	.75	.81	.75	.66
	5	.95	.92	.90	.89	.88	.89	.88	.83
5	15	.97	.97	.96	.94	.93	.95	.94	.93
	20	.98	.97	.96	.96	.95	.96	.95	.94
	25	.98	.98	.96	.96	.97	.97	.96	.95
	1	.91	.86	.83	.79	.76	.82	.75	.65
	5	.95	.92	.90	.89	.87	.91	.88	.85
10	15	.97	.96	.95	.94	.93	.95	.94	.93
	20	.97	.97	.96	.96	.95	.96	.95	.95
	25	.98	.97	.97	.97	.96	.96	.96	.96
	1	.91	.87	.82	.81	.77	.81	.75	.65
	5	.94	.91	.90	.89	.88	.91	.89	.85
20	15	.97	.95		.94	.94	.95	.95	.92
	20	.97	.96	.96	.95	.95	.96	.95	.95
	25	.98	.97	.97	.96	.96	.97	.96	

BIBLIOGRAPHY

1. Akaike, Hirotugu, "On Ergodic Property of a Tandem Type Queueing Process," Annals of the Institute of Statistical Mathematics (Japan), Vol. 9 (1957), pp. 13-21.

2. Bibliography on Simulation, IBM Manual 320-0924-0, Technical Publication Department, White Plains, N. Y., 1964.

3. Burke, Paul J., "The Output of a Queueing System," Operations Research, Vol. 4 (1956), pp. 699-704.

4. Burke, P. J. "The Dependence of Delays in Tandem Queues," Annals of Mathematical Statistics, Vol. 35 (1964), pp. 874-875.

5. Chesbrough, P. E., "The Output of Queueing Systems," Thesis, Georgia Institute of Technology, 1966, 113 pp.

6. Chang, Wei, "Output Distribution of a Single-Channel Queue," Operations Research, Vol. 11 (1963), pp. 620-623.

7. Cohen, J. W., "On the Queueing Process of Lanes, " Philips Technical Report, 1956.

8. Conolly, B. W., "A Difference Equation Technique Applied to the Simple Queue," Journal of the Royal Statistical Society, Series B, Vol. 20 (1958), pp. 165-167.

9. Conolly, B. W., "A Difference Equation Technique Applied to the Simple Queue with Arbitrary Interval Distribution," Journal of the Royal Statistical Society, Series B, Vol. 20 (1958), pp. 167-175.

10. Conolly, B. W., "Queueing at a Single Serving Point with Group Arrival," Journal of the Royal Statistical Society, Series B, Vol. 22 (1960), pp. 258-298.

11. Conway, R. W., and Maxwell, "A Queueing Model with State Dependent Service Rate," Journal of Industrial Engineering, Vol. 12, No. 2, March-April 1961, pp. 132-156.

12. Conway, R. W., Maxwell, W. L., and Miller, L. W., Theory of Scheduling, Addison-Wesley Publishing Company, Reading, Massachusetts- Pala Alto-London-Don Mills, Ontario, 1967, x+293 pp.

13. Cox, D. R., "Some Statistical Methods Connected with Series of Events," Journal of the Royal Statistical Society, Series B, Vol. 17 (1955), pp. 129-164.

14. Cox, D. R., and W. L. Smith, Queues, Methuen's Monographs on Statistical Subjects, Methuen and Co., Ltd., London; John Wiley and Sons, Inc., New York; 1961, xii+180 pp.

15. DeBaun, R. M., and S. Katz, "An Approximation to Distribution of Summed Waiting Times," Operations Research, Vol. 7 (1959), pp. 811-813.

16. Digital Simulation in Operation Research, A Conference Under the Aegis of the Scientific Affairs Division of N.A.T.O., American Elsevier Publishing Company, Inc., 392 pp.

17. de la Cruz, J., and J. Tolra, "An Application of the Monte-Carlo Method to the Study of Assembly Lines, " Eighth International Meeting of the Institute of Management Sciences, Brussels, Belguin, Aug. 1961.

18. Doig, A., "A Bibliography on the Theory of Queues," Biometrika, Vol. 44 (1957), pp. 490-514.

19. Dunn, Paul F., C. D. Flagle, and P. A. Hicks, "The Queuiac: An Electromechanical Analog for the Simulation of Waiting-Line Problems," Operations Research, Vol. 4 (1956), pp. 648-662.

20. Finch, P. D., "The Effect of the Size of the Waiting Room on a Simple Queue," Journal of the Royal Statistical Society, Series B, Vol. 20 (1958), pp. 182-186.

21. Finch, P. D., "The Output Process of the Queueing System M/G/1," Journal of the Royal Statistical Society, Series B, Vol. 21 (1959), pp. 375-380.

22. Ghosal, A., "Queues in Series," Journal of the Royal Statistical Society, Series B, Vol. 24 (1962), pp. 359-363.

23. Good, I. J., "The Number of Individuals in a Cascade Process," Proceedings of the Cambridge Philosophical Society, Vol. 45 (1948), pp. 360-363.

24. Hammersley, J. M., and D. C. Handscomb, Monte-Carlo Methods, Methuen and Co., Ltd., London, and John Wiley and Sons, Inc., New York, 1964, iv+178

25. Harris, C. M., "Queues with state dependent stochastic service rates," Operations Research, Vol. 15, Pt 1, 1967, pp. 117-130.

26. Hillier, F. S., and R. W. Boling, "Finite Queues in Series with Exponential or Erlang Service Times - A Numerical Approach," Operations Research, Vol. 15, Part 1, Jan-Aug 1967, pp. 287-303.

27. Hunt, G. C., "Sequential Arrays of Waiting Lines," Operations Research, Vol. 4 (1956), pp. 674-683.

28. IBM, General Purpose Systems Simulator II, Reference Manual, IBM Data Processing Division, 1963.

29. Jackson, J. R., "Multiple Servers with Limited Waiting Space," Naval Research Logistics Quarterly, Vol. 5 (1958), pp. 315-321.

30. Jackson, J. R., Networks of Waiting Lines," Operations Research, Vol. 5 (1957), pp. 518-521.

31. Jackson, R. R. P., "Queueing Systems with Phase-Type Service," Operations Research Quarterly, Vol. 5 (1954), pp. 109-120.

32. Jackson, R. R. P., "Random Queueing Process with Phase-Type Service," Journal of the Royal Statistical Society, Series B, Vol. 18 (1956), pp. 129-132.

33. Lindley, D. V., "The Theory of Queues with a Single Server," Proceedings of the Cambridge Philosophical Society, Vol. 48 (1952), pp. 277-289.

34. Loynes, R. M., "The Stability of a System of Queues with a Single Server," Proceedings of the Cambridge Philosophical Society, Vol. 58 (1962), pp. 497-500.

35. Lunger, G. F., "Bibliography on Queueing Theory," Sperry Rand Corporation, Univac Division, 1959.

36. Makino, Toji, "Tandem-Gate Machi-Gyoretsu Ni San No Mondai," ("Some Problems on Tandem Queues"), Management Science (Japan), Vol. 6 (1963), pp. 246-257.

37. Makino, Toji, "On the Mean Passage Time Concerning Some Queueing Problems of the Tandem Type," Journal of the Operations Research Society of Japan, Vol. 7 (1964), pp. 17-47.

38. Malcolm, D. G., "Bibliography on the Use of Simulation in Management Analysis," Operations Research, Vol. 8, No. 2, pp. 169-177.

39. Marshal, A. W., and E. Reich, "Study of the Characteristics of Queues in Tandem," Operations Research, Vol. 4 (1956), P. 386.

40. Masterson, Gregory E., and Seymour Sherman, "On Queues in Tandem," Annals of Mathematical Statistics, Vol. 34 (1963), pp. 300-307.

41. McMillan, C., and Gonzalez, Systems Analysis, Richard D. Irwin, Inc., Homewood, Illinois, 1965, xiii+333pp.

42. Mize, Joe H., and J. Grady Cox, Essentials of Simulation, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1968, xiii+233 pp.

43. Morris, W. T., Analysis for Materials Handling Management, Richard D., Inc., Homewood, Illinois, 1962, ix+204 pp.

44. Naylor, T. H., J. L. Balintfy, D. S. Burdick, and K. Chu, Computer Simulation Techniques, John Wiley and Sons, Inc., New York, 1966, x+352 pp.

45. Nelson, Rosser T., "Waiting Time Distribution for Application to a Series of Service Centers," Operations Research, Vol. 6 (1958), pp. 856-862.

46. Nelson, Rosser T., "Queueing Network Experiments with Varying Arrival and Service Processes," Naval Research Logistics Quarterly, Vol. 13, No. 3, September 1966, pp. 321-347.

47. O'Brian, G. G., "The Solution of Some Queueing Problems," Journal of the Society for Industrial and Applied Mathematics, Vol. 2 (1954), pp. 133-142.

48. Pala, E. R., C. A. Beloso, and W. W. Hines, An Introduction to the Theory and Application of Waiting Line Models, Reinhold Publishing Corporation, New York, 1967, x+180.

49. Reich, E., "Birth-Death Process and Tandem Queues," The Rand Corporation, Paper, P. 863, 1956.

50. Reich, E., "Waiting Times When Queues are in Tandem," Annals of Mathematical Statistics, Vol. 28 (1957), pp. 768-773.

51. Reich, E., "Note on Queues in Tandem," Annals of Mathematical Statistics, Vol. 34 (1963), pp. 338-341.

52. Saaty, Thomas L., Elements of Queueing Theory with Applications, McGraw-Hill Book Company Inc., New York-Tornnto-London, 1961, xv+423 pp.

53. Saaty, Thomas L., "Seven More Years of Queues, A Lament and a Bibliography," Naval Research Logistics Quarterly, Vol. 13, No. 4, December 1966, pp. 447-476.

54. Sacks, J., "Ergodicity of Queues in Series," Annals of Mathematical Statistics, Vol. 31 (1960), pp. 579-588.

55. Sasieni, M., A. Yaspan, and L. Friedman, Operations Research, Methods and Problems, John Wiley and Sons, Inc., New York-London, 1959, pp. 125-154.

56. Shubik, Martin, "Bibliography on Simulation, Gaming, Artificial Intelligence and Allied Topics," Journal of the American Statistical Association, Vol. 55, 1960, pp. 736-751.

57. Suzuki, Takeji, "Two Queues in Series," Journal of the Operations Research Society of Japan, Vol. 5 (1962/1963), pp. 125-154.

58. Takacs, L., Introduction to the Theory of Queues, University Texts in the Mathematical Sciences, Oxford University Press, New York, 1962, x+268 pp.

59. Tocher, K. D., The Art of Simulation, D. Van Nostrand, Princeton, New Jersey, 1963, viii+184.

60. Torgerson, P. E., H. E. Hays and A. Abruzi, "Introducing Queueing Concepts: A Simulation Approach," Journal of Industrial Engineering, May 1967, p. 328.